16. Counting: finite sets

One way to use sets and functions is to count. We will first focus on counting finite sets.

Proposition 16.1 (Inclusion-Exclusion). *Let A, B and C be three finite sets.*

Then

$$|A\cup B\cup C| = |A|+|B|+|C|-|B\cap C|-|A\cap C|-|A\cap B|+|A\cap B\cap C|.$$

Proof. We sketch a proof of this result; a formal proof is part of the seventh homework.

We first do the case of two sets. How many elements belong to $A\cup B$? As a first approximation, we might say that the answer is $|A| + |B|$. The problem is that the elements of $A \cap B$ get counted twice. Once as elements of A and once as elements of B. Since we included the elements of $A \cap B$ one time too many, we need to exlcude them. So the formula is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Now consider how the number of elements of $A \cup B \cup C$. As a first approximation, we might say that the answer is $|A| + |B| + |C|$. The problem is that the elements of $A\cap B$, $A\cap C$ and $B\cap C$ all get counted twice. So we need to exclude these elements. How many times do we then count an element of $A\cap B\cap C$? Three times as elements of A, B and C. But then we exclude them three times as elements of $A \cap B$, $A \cap C$ and $B \cap C$. So we need to include $A \cap B \cap C$ one more time.

So the formula is

$$|A\cup B\cup C| = |A|+|B|+|C|-|B\cap C|-|A\cap C|-|A\cap B|+|A\cap B\cap C|. \quad \square$$

Question 16.2. *Consider a chessboard. Starting at the bottom left corner, how many ways are there of getting to the top right hand corner, if at every stage you can either go right or go up?*

Introduce a function

$$f\colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N},$$

defined by the rule:

$$f(i, j) = \text{the number of ways to get to square } (i, j).$$

It is easy to fill in some of the numbers. If $j = 0$ then there are no choices, you just have to keep going right,

$$f(i, 0) = 1.$$

If $i = 0$ there are again no choices, you can only go up,

$$f(0, j) = 1.$$

In fact, by symmetry,

$$f(i, j) = f(j, i).$$

For every way to get from $(0, 0)$ to (i, j) there is a mirror symmetric way to get from $(0, 0)$ to (j, i). So we only need to figure out $f(i, j)$ for $j \leq i$. The most important thing of all is that there is a recursive formula for $f(i, j)$:

Lemma 16.3. *If* $(i, j) \in \mathbb{N} \times \mathbb{N}$ *then*

$$f(i + 1, j + 1) = f(i, j + 1) + f(i + 1, j).$$

Proof. This is in fact very simple. If we go from $(0, 0)$ to $(i + 1, j + 1)$ then at the very last step either we are at $(i, j+1)$ or we are at $(i+1, j)$. So the number of ways to get from $(0, 0)$ to $(i+1, j+1)$ is equal to the number of ways to get to $(i + 1, j)$ plus the number of ways to get to $(i, j + 1)$. □

Using this we can figure out the number of ways to get to square $(7, 7)$, that is, compute $f(7, 7)$, at least in practice (3432).

There is another way to think about all of this. Instead of keeping track of the position of the square we want to move to, instead keep track of the number of steps we take and the number of times we go right (or East). Let $n = i + j$, so that n is the total number of steps (that is, the number of squares we visit, minus one). To get to square (i, j) we have to make n decisions. At each square we have to decide to go right or up. Of those n times, we have to decide to go right i times. Think of the n decision times are being n objects; from those n objects we have to pick, or designate right, i objects.

Let $U \subset \mathbb{N} \times \mathbb{N}$ be the subset of all pairs (n, k) where $k \leq n$. Let

$$g \colon U \longrightarrow \mathbb{N},$$

be the function defined by

$$g(n, k) = f(k, n - k).$$

Then

$$g(n, k) = \text{number of ways to pick } k \text{ objects from } n \text{ objects.}$$

Definition 16.4. *Let n and k be two natural numbers, with $n \geq k$. n* ***choose*** *k, denoted*

$$\binom{n}{k}$$

is the number of ways to pick k objects from n objects.

Proposition 16.5. *Let n and k be two natural numbers, with $n \geq k$. Then*

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}.$$

Proof. We give two proofs of this result. We will give a third later. We have already observed that

$$\binom{n}{k} = g(n, k).$$

We have

$$\begin{aligned}
\binom{n+1}{k+1} &= g(n+1, k+1) \\
&= f(k+1, n-k) \\
&= f(k+1, n-k-1) + f(k, n-k) \\
&= g(n, k+1) + g(n, k) \\
&= \binom{n}{k+1} + \binom{n}{k}.
\end{aligned}$$

Aliter: For the second proof, we imagine colouring one of the $n+1$ objects red and all the other objects are blue. There are two ways to pick $k+1$ objects from the $n+1$ objects. Either we include the red object or we don't. If we include the red object, we just have to pick a further k blue objects. There are

$$\binom{n}{k}$$

ways to do this. Or we don't include the red object. But then we need to pick $k+1$ of the blue objects from the n blue objects. There are

$$\binom{n}{k+1}$$

ways to do this. □

There is another way to think of all of this. Imagine describing a path from $(0, 0)$ to (i, j) by a string of letters, let's say R and U. So

$$RRURUUU$$

means go right, go right, go up, go right, go up, go up, go up. This describes one path from $(0, 0)$ to $(3, 4)$. All paths from $(0, 0)$ to $(3, 4)$ are described by all strings of R and U, using three R's and four U's.

Theorem 16.6 (Binomial Theorem)**.** *If n is an integer and x and y are two variables then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{i+j=n} \binom{i+j}{i} x^i y^j.$$

Proof. We have to expand

$$(x+y)(x+y)\dots(x+y).$$

If we do so we get all strings of n letters consisting of x's and y's. If we group together all strings which use i x's and $j = n-i$ y's then on the one hand we are computing the coefficient of $x^i y^j$ and on the other hand we are counting the number of ways to get from $(0,0)$ to (i,j). □

Theorem 16.7. *Let k and n be two natural numbers, with $n \geq k$. Then*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof. We want to count the number of ways to pick k objects from n objects.

We first count the number of ways to pick k objects in order. There are n ways to pick the first object, $n-1$ ways to pick the second object down to $n-k+1$ to pick the last object. The formula is then

$$n(n-1)(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!}.$$

But this counts the number of ways to pick k objects from n objects in order. There are $k!$ ways to order k objects, so

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$ □

This gives us yet another:

Proof of (16.5). We have

$$\begin{aligned}\binom{n}{k+1} + \binom{n}{k} &= \frac{n!}{(k+1)!(n-k-1)!} + \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k-1)!}\left(\frac{1}{k+1} + \frac{1}{n-k}\right) \\ &= \frac{n!}{k!(n-k-1)!}\left(\frac{(n-k)+(k+1)}{(k+1)(n-k)}\right) \\ &= \frac{n!}{k!(n-k-1)!}\left(\frac{(n+1)}{(k+1)(n-k)}\right) \\ &= \frac{(n+1)!}{(k+1)!(n-k)!} \\ &= \binom{n+1}{k+1}.\end{aligned}$$

□

We recall one more feature of counting finite sets. We already observed that if A and B are sets and $f\colon A \longrightarrow B$ is an injective function then $|A| \leq |B|$, where the cardinality is taken the naive sense (either a finite or infinite). The converse of this observation is surprisingly useful:

Proposition 16.8 (Pigeonehole principle)**.** *Let A and B be two sets and let $f\colon A \longrightarrow B$ be a function.*

If $|A| > |B|$ then f is not injective. In particular there is an element b of B and two distinct elements a_1 and a_2 of A such that $f(a_1) = b = f(a_2)$.

Informally if you have more pigeons than pigeonholes and you put the pigeons into the pigeonholes then there is at least one pigeonhole with at least two pigeons.