6. Odds and Ends

Here we collect a couple of ad hoc techniques.

Even though proof by contradiction is a useful technique, it is often possible to avoid proof by contradiction using a different technique to prove the same result.

One way to prove

$$p \implies q$$

is to prove the contrapositive

$$\neg q \implies \neg p.$$

In fact both statements are equivalent. The first statement is only false if p is true and q is false. The second statement is only false if $\neg q$ is true and $\neg p$ is false, which is to say, q is true and p is false, the same as before.

Lemma 6.1. *Let n be an integer.*

If $n^2 - 6n + 5$ is even then n is odd.

Proof. It suffices to prove the contrapositive, that if n is even then $n^2 - 6n + 5$ is odd.

Suppose n is even. Then there is an integer m such that $n = 2m$. In this case

$$\begin{aligned} n^2 - 6n + 5 &= (2m)^2 - 6(2m) + 5 \\ &= 4m^2 - 12m + 4 + 1 \\ &= 2(2m^2 - 6m + 2) + 1. \end{aligned}$$

As $2m^2 - 6m + 2$ is an integer it follows that $n^2 - 6n + 5$ is odd. □

The next observation is that if you want to know if something is true or false, it is often sufficient to give a single example, often called a counterexample, especially if you are trying to show the statement is false.

Question 6.2. *True or false?*

If n is an integer then

$$n(n+1)$$

is always divisible by 3.

False.

Take $n = 1$. Then

$$n(n+1) = 1 \cdot 2 = 2,$$

which is not divisible by 3, as $2 < 3$. Thus $n = 1$ is a counterexample to the statement that

$$n(n+1)$$

is always divisible by 3.

Finally, sometimes to prove a statement, it helps to go backwards from the goal, instead of forwards from the hypotheses. This is often the case for inequalities.

Theorem 6.3. *If a and b are any real numbers then*

$$a^2 + b^2 \geq 2ab.$$

Proof. As $a - b$ is a real number and the square of any real number is non-negative we have,

$$(a-b)^2 \geq 0.$$

Expanding we get

$$a^2 - 2ab + b^2 \geq 0,$$

so that

$$a^2 + b^2 \geq 2ab. \qquad \square$$