MODEL ANSWERS TO THE THIRD HOMEWORK

1. We prove this result by mathematical induction. Let $P(n)$ be the statement that

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

We have to prove that $P(1)$ holds and that $P(k) \implies P(k+1)$ for any positive integer k.
If $n = 1$ then the LHS is

$$1^2 = 1,$$

and the RHS is

$$\frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)2+1}{6} = 1.$$

As both sides are equal, it follows that $P(1)$ is true.
Now suppose that k is a positive integer and $P(k)$ holds. We check that $P(k+1)$ holds.
We have

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= [1^2 + 2^2 + 3^2 + \cdots + k^2] + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6},
\end{aligned}$$

where we use the fact that $P(k)$ is true to get from the first line to the second line. It follows that $P(k+1)$ holds.
As we have shown that $P(1)$ is true and that $P(k) \implies P(k+1)$ for any positive integer k, by the principle of mathematical induction it follows that $P(n)$ holds for all n, that is

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. (5.1) We prove this result by mathematical induction. Let $P(n)$ be the statement that

$$n^3 - n$$

is divisible by 3.
We have to prove that $P(1)$ holds and that $P(k) \implies P(k+1)$ for any positive integer k.
If $n = 1$ then

$$n^3 - n = 1^3 - 1 = 0,$$

is divisible by 3. Thus $P(1)$ holds.
Now suppose that k is a positive integer and $P(k)$ holds. We check that $P(k+1)$ holds.
We have

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3(k^2 + k)\end{aligned}$$

Now $k^3 - k$ is divisible by 3 by induction and $3(k^2+k)$ is divisible by 3 by inspection, as k^2+k is an integer. Thus $(k+1)^3-(k+1)$ is divisible by 3 and so $P(k+1)$ is true.
As we have shown that $P(1)$ is true and that $P(k) \implies P(k+1)$ for any positive integer k, by the principle of mathematical induction it follows that $P(n)$ holds for all n, that is

$$n^3 - n,$$

is divisible by 3.
(5.2) We prove this result by mathematical induction. Let $P(n)$ be the statement that

$$n^3 \leq 2^n.$$

We have to prove that $P(10)$ holds and that $P(k) \implies P(k+1)$ for any integer $k \geq 10$.
If $n = 10$ then the LHS is

$$n^3 = 10^3 = 2^3 \cdot 5^3$$

and the RHS is

$$2^n = 2^{10} = 2^3 \cdot 2^7.$$

Now $5^3 = 125$ and $2^7 = 128$ thus

$$\begin{aligned}10^3 &= 2^3 \cdot 5^3 \\ &\leq 2^3 \cdot 2^7 \\ &= 2^{10}.\end{aligned}$$

Thus $P(10)$ holds.

Now suppose that k is a positive integer and $P(k)$ holds. We check that $P(k+1)$ holds.
We have

$$\begin{aligned}(k+1)^3 &= k^3+3k^2+3k+1\\ &\le k^3+3k^2+3k^2+3k^2\\ &= k^3+9k^2\\ &\le k^3+k^3\\ &= 2k^3\\ &< 2\cdot 2^k\\ &\le 2^{k+1},\end{aligned}$$

where we used the fact that $3k \le 3k^2$ and the fact that $1 \le 3k^2$, for $k \ge 1$, to get from the first line to the second line; we used the fact that $9k^2 \le k^3$, for $k \ge 10$, to get from the third line to the fourth line and the inductive hypothesis to get from the fifth to the sixth line. Thus $P(k+1)$ holds.
As we have shown that $P(10)$ is true and that $P(k) \implies P(k+1)$ for any integer $k \ge 10$, by the principle of mathematical induction it follows that $P(n)$ holds for all $n \ge 10$, that is

$$n^3 \le 2^n.$$

(5.3) We prove this result by mathematical induction. Let $P(n)$ be the statement that $n \ge 1$.
We have to prove that $P(1)$ holds and that $P(k) \implies P(k+1)$ for any positive integer k.
If $n = 1$ then the LHS is 1 and as $1 \ge 1$ is a true statement, $P(1)$ is true.
Now suppose that $P(k)$ is true. We have

$$\begin{aligned}k+1 &\ge k\\ &\ge 1,\end{aligned}$$

where we used $P(k)$ to get from the first line to the second line. Thus $P(k+1)$ holds.
As we have shown that $P(1)$ is true and that $P(k) \implies P(k+1)$ for any positive integer k, by the principle of mathematical induction it follows that $P(n)$ holds for all n, that is

$$n \ge 1.$$

(5.4) We prove this result by mathematical induction. Let $P(n)$ be the statement that

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

We have to prove that $P(0)$ holds and that $P(k) \implies P(k+1)$ for any non-negative integer k.

If $n = 0$ the LHS is 1 and the RHS is also

$$\frac{1 - x^{n+1}}{1 - x} = \frac{1 - x}{1 - x} = 1.$$

Thus $P(0)$ holds.

Now suppose that $P(k)$ holds. We check that $P(k+1)$ holds.

$$\begin{aligned}
1 + x + x^2 + \cdots + x^k + x^{k+1} &= [1 + x + x^2 + \cdots + x^k] + x^{k+1} \\
&= \frac{1 - x^{k+1}}{1 - x} + x^{k+1} \\
&= \frac{1 - x^{k+1} + (1 - x)x^{k+1}}{1 - x} \\
&= \frac{1 - x^{k+1} + x^{k+1} - x^{k+2}}{1 - x} \\
&= \frac{1 - x^{k+2}}{1 - x},
\end{aligned}$$

where we used $P(k)$ to get from the first line to the second line. Thus $P(k+1)$ holds.

As we have shown that $P(0)$ is true and that $P(k) \implies P(k+1)$ for any non-negative integer k, by the principle of mathematical induction it follows that $P(n)$ holds for all n, that is

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

3. (i)

$$1, 2, 3/2, 5/3, 8/5, \ldots.$$

(ii) We prove this result by mathematical induction. Let $P(n)$ be the statement that

$$b_n = \frac{F_{n+1}}{F_n}.$$

We have to prove that $P(1)$ holds and that $P(k) \implies P(k+1)$ for any positive integer k.

We check that $P(1)$ holds. If $n = 1$ then $b_1 = 1$ and the RHS is equal to

$$\frac{F_{n+1}}{F_n} = \frac{1}{1} = 1.$$

Thus $P(1)$ holds.
Now suppose that $P(k)$ holds. We check that $P(k+1)$ holds.

$$\begin{aligned} b_{k+1} &= 1 + \frac{1}{b_k} \\ &= 1 + \frac{1}{\frac{F_k}{F_{k-1}}} \\ &= 1 + \frac{F_{k-1}}{F_k} \\ &= \frac{F_{k-1} + F_k}{F_k} \\ &= \frac{F_{k+1}}{F_k}, \end{aligned}$$

where we used the inductive hypothesis to get from the first line to the second line. Thus $P(k+1)$ holds.
As we have shown that $P(1)$ is true and that $P(k) \implies P(k+1)$ for any positive integer, $P(n)$ is true for all positive integers, that is

$$b_n = \frac{F_{n+1}}{F_n}.$$

(iii) We prove this result by mathematical induction. Let $P(n)$ be the statement that

$$b_{n+1} - b_n = \frac{(-1)^{n+1}}{F_n F_{n+1}}.$$

We have to prove that $P(1)$ holds and that $P(k) \implies P(k+1)$ for any positive integer k.
We check that $P(1)$ holds. If $n = 1$ then the LHS is

$$\begin{aligned} b_{n+1} - b_n &= b_2 - b_1 \\ &= 2 - 1 \\ &= 1, \end{aligned}$$

and the RHS is equal to

$$\frac{(-1)^{n+1}}{F_n F_{n+1}} = \frac{(-1)^2}{F_1 F_2} = \frac{1}{1} = 1.$$

Thus $P(1)$ holds.

Now suppose that $P(k)$ holds. We check that $P(k+1)$ holds. We have

$$\begin{aligned} b_{k+2} - b_{k+1} &= (1 + \frac{1}{b_{k+1}}) - (1 + \frac{1}{b_k}) \\ &= \frac{1}{b_{k+1}} - \frac{1}{b_k} \\ &= \frac{b_k - b_{k+1}}{b_k b_{k+1}} \\ &= -\frac{b_{k+1} - b_k}{\frac{F_{k+1}}{F_k}\frac{F_{k+2}}{F_{k+1}}} \\ &= -\frac{\frac{(-1)^{k+1}}{F_k F_{k+1}}}{\frac{F_{k+2}}{F_k}} \\ &= -\frac{(-1)^{k+1}}{F_k F_{k+1}} \frac{F_k}{F_{k+2}} \\ &= \frac{(-1)^{k+2}}{F_{k+1} F_{k+2}}, \end{aligned}$$

where we used the recursive definition of the sequence for the first line, (i), twice, to get from the third line to the fourth line and the inductive hypothesis to get from the fourth line to the fifth line. Thus $P(k+1)$ holds.

As we have shown that $P(1)$ is true and that $P(k) \implies P(k+1)$ for any positive integer, $P(n)$ is true for all positive integers, that is

$$b_{n+1} - b_n = \frac{(-1)^{n+1}}{F_n F_{n+1}}.$$

4. The trick is to write down the first five cases by hand and then use the fact that we can get any other postage by adding an appropriate number of five cent stamps.

Formally we want to show that we can solve the Diophantine equation

$$5x + 9y = n,$$

as soon as $n \geq 34$, using non-negative integers x and y.

Let $P(n)$ be the statement that we can find non-negative integer solutions of the equation

$$5x + 9y = n.$$

We will prove that $P(n)$ holds for all integers $n \geq 34$ using strong mathematical induction.

We first check that $P(34)$, $P(35)$, $P(36)$, $P(37)$ and $P(38)$ hold:

In fact $(x, y) = (5, 1)$, $(7, 0)$, $(0, 4)$, $(2, 3)$ and $(4, 2)$ are non-negative solutions to the equations:

$$5x+9y = 34 \quad 5x+9y = 35 \quad 5x+9y = 36 \quad 5x+9y = 37 \quad \text{and} \quad 5x+9y = 38,$$

Thus $P(34)$, $P(35)$, $P(36)$, $P(37)$ and $P(38)$ are true.
We now check that if $k \geq 38$ and $P(j)$ holds for all $34 \leq j \leq k$ then $P(k + 1)$ holds.
Let $j = k + 1 - 5 \leq k$. Note that

$$\begin{aligned} j &= k + 1 - 5 \\ &= k - 4 \\ &\geq 38 - 4 \\ &\geq 34. \end{aligned}$$

By our inductive hypothesis, $P(j)$ holds, so that we may find non-negative integers x and y such that

$$5x + 9y = j.$$

Then $(x + 1, y)$ are non-negative integers and

$$\begin{aligned} 5(x + 1) + 9y &= 5x + 5 + 9y \\ &= 5x + 9y + 5 \\ &= j + 5 \\ &= k + 1. \end{aligned}$$

Thus $P(k + 1)$ holds.
As we have shown that $P(34)$, $P(35)$, $P(36)$, $P(37)$ and $P(38)$ are true and that if $k \geq 38$ then $P(j)$ for all $34 \leq j \leq k$ implies $P(k + 1)$, we know that $P(n)$ holds for all $n \geq 34$ by strong mathematical induction, that is, we can solve the Diophantine equation

$$5x + 9y = n,$$

as soon as $n \geq 34$, using non-negative integers x and y.
Challenge problems/Just for fun:
5. This is much easier than it looks. Let $P(n)$ be the statement that we can find a successful trip, with $2n$ dots.
We prove this by mathematical induction. $P(0)$ is obviously true, since there are no dots at all.
Suppose that $P(k)$ is true. Suppose we are presented with a colouring with $2(k + 1)$ colours. Look for a pair consisting of a red dot followed by a blue dot, in that order, going clockwise around the circle. A moment's thought will convince you that such a pair always exists;

just start anywhere and wait until you cross a red dot. Now wait until you cross a blue dot.
Consider the $2k$ remaining dots. By induction we know there is somewhere to start so that we always cross at least as many red dots as blue dots. We can always move this starting point, so that it is not in the middle of the two consecutive red and blue dots we are ignoring.
Now consider what happens as we go around the circle. Until we cross the two special dots, we meet at least as many red dots as blue dots. When we meet one of the two special dots, we will first meet the red dot, by our choice of special dots and our choice of starting point. After we cross the red dot then we cross the blue dot. After this, we continue to meet at least as many red dots as blue dots.
6. The case $p = 0$ is easy, since $n^0 = 1$, and so the sum is n. The case $p = 1$ is done in lectures. The case $p = 2$ is the first problem. We will do the case when $p = 3$ (I am not very patient).
We guess the sum is given by a polynomial of degree 4. If $n = 0$ then the sum is zero, and so this quartic is divisible by n. Therefore

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = n(an^3 + bn^2 + cn + d).$$

Let's compute a couple of these sums. If $n = 1, 2, 3, 4, 5, 6$, we get

$$1 \qquad 9 \qquad 36 \qquad 100 \qquad 225 \qquad \text{and} \qquad 441.$$

Note that the first four numbers are squares. In fact $225 = 15^2$ and $441 = 21^2$.
So we might guess that the RHS is a square.

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = n^2(an + b)^2.$$

There is not much for it but to pick two values of n and use this to compute a and b. When $n = 1$ and 2,

$$1 = (a + b)^2 \qquad \text{and} \qquad 9 = 4(2a + b)^2.$$

If we guess a and b are non-negative we get

$$1 = a + b \qquad \text{and} \qquad 3 = 2(2a + b).$$

It follows that $a = 1/2$ and $b = 1/2$. Thus we guess

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Let $P(n)$ be the statement that this is correct. We prove $P(n)$ holds for all non-negative integers by induction.
If $n = 1$ the LHS is

$$1^3 = 1$$

and the RHS is

$$\left(\frac{n(n+1)}{2}\right)^2 = \left(\frac{1(1+1)}{2}\right)^2 = 1.$$

Thus $P(1)$ holds.
We now check that $P(k+1)$ holds if $P(k)$ holds.

$$\begin{aligned} 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= [1^3 + 2^3 + \cdots + k^3] + (k+1)^3 \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2(k^2+4k+4)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \left(\frac{(k+1)(k+2)}{2}\right)^2. \end{aligned}$$

Thus $P(k+1)$ holds.
As we checked that $P(1)$ holds and that $P(k) \implies P(k+1)$, by the principle of mathematical induction, $P(n)$ holds for all n, that is,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

7. Not much; clearly this result is nonsense.
The problem happens when there are two cows, that is, the problem is with the implication $P(1) \implies P(2)$. Let's suppose one of the cows is brown and the other is grey. We pick the grey cow and put it to one side (metaphorically speaking). It is true that the one remaining cow is monochromatic.
The problem comes at the next stage. We are supposed to form a group of $k = 1$ cows, from the cow we put to one side and any $k - 1$ of the brown cows. It is again true that all of the cows have the same colour, now grey. The problem is that $k - 1 = 0$ so that none of the group of k cows is brown.
It is interesting to note that this is the only place where the argument breaks down. In other words, if you have n objects and you know that any pair of objects has the same colour then all n objects have the same colour. The proof is the one sketched in question 7, the only difference is we now start the induction at $n = 2$.

8. There is clearly a problem with this argument. Somehow the issue is with the notion of what constitutes a "surprise", but it is hard to exactly pinpoint the problem. In fact philosophers like to debate this issue, at length.