MODEL ANSWERS TO THE FOURTH HOMEWORK

1. Let $P(n)$ be the statement that

$$(1+x)^n \geq 1 + nx.$$

We want to prove that $P(n)$ holds for all non-negative n. We proceed by mathematical induction.

We first check that $P(0)$ holds. If $n = 0$ the LHS is

$$(1+x)^n = (1+x)^0 = 1,$$

and the RHS is

$$1 + nx = 1.$$

As the LHS is at least the RHS, $P(0)$ holds.

Now assume that $P(k)$ holds. We check that $P(k+1)$ holds. We have

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) \\
&\geq (1+kx)(1+x) \\
&= 1 + kx + x + x^2 \\
&= 1 + (k+1)x + x^2 \\
&\geq 1 + (k+1)x,
\end{aligned}$$

where we got from the first line to the second line using the inductive hypothesis $P(k)$. Thus $P(k+1)$ holds.

As we checked that $P(0)$ holds and that $P(k) \implies P(k+1)$ for every k, by the principle of mathematical induction it follows that $P(n)$ holds for every n, that is,

$$(1+x)^n \geq 1 + nx.$$

2. (a)

$$\emptyset, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4, \}, \{1, 2, 3, 4\}.$$

(b)

$$\{1\}, \{2\}, \{3\}, \{4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}.$$

3. (a) True.

$$\{x \in \mathbb{R} \mid x > 0, (x^2 - 1)^2 = 0\} = \{1\}$$

so that

$$\{\{1\}, \{x \in \mathbb{R} \mid x > 0, (x^2 - 1)^2 = 0\}\}$$

has one element, $\{1\}$.

(if we changed things slightly, and we looked at

$$|\{\,\{\,1\,\},\{\,x\in\mathbb{R}\,|\,x>0,(x^2+1)^2=0\,\}\,\}|=1.$$

we would get a false statement.

$$\{\,x\in\mathbb{R}\,|\,x>0,(x^2-1)^2=0\,\}=\emptyset,$$

so that

$$\{\,\{\,1\,\},\{\,x\in\mathbb{R}\,|\,x>0,(x^2-1)^2=0\,\}\,\}$$

has two elements, $\emptyset$ and $\{\,1\,\}$.)
(b) False.

$$\{\,\emptyset\,\}$$

is the set containing one element, the emptyset. The set

$$\{\,\{\,\emptyset\,\},2\,\}.$$

does not contain the emptyset (even though it contains a set that contains the emptyset).
(c) True.
The last set

$$\{\,x\in\mathbb{R}\,|\,x^2\geq 0\,\}$$

is the set of all real numbers, $\mathbb{R}$. So the set

$$\{\,1,\mathbb{R},\{\,x\in\mathbb{R}\,|\,x^2\geq 0\,\}\,\}$$

has two, elements 1 and $\mathbb{R}$.
4. (a) We have

$$\begin{aligned} A\bigtriangleup\emptyset &= A\cup\emptyset\setminus A\cap\emptyset \\ &= A\setminus\emptyset \\ &= A. \end{aligned}$$

(b) We have

$$\begin{aligned} A\bigtriangleup A &= A\cup A\setminus A\cap A \\ &= A\setminus A \\ &= \emptyset. \end{aligned}$$

(c) There are two ways to prove this.
The first is to prove this element by element. We first show that $B\subset C$. Suppose that $b\in B$. We have to show that $b\in C$. There are two cases. Suppose that $b\notin A$. Then $b\in B\setminus A$ so that $b\in A\bigtriangleup B$. Thus $b\in A\bigtriangleup C$. As $b\notin A$, $b\notin A\setminus C$ so that $b\in C\setminus A$. It follows that $b\in C$.
Now suppose that $b\in A$. Then $b\notin A\setminus B$ and $b\notin B\setminus A$ so that $A\bigtriangleup B$. Therefore $b\notin A\bigtriangleup C$. In particular $b\notin A\setminus C$. As $b\in A$ it follows that $b\in C$.

Either way, $b \in C$ and so $B \subset C$. By symmetry $C \subset B$. It follows that $B = C$.
Aliter:
We have

$$\begin{aligned} B &= B \triangle \emptyset \\ &= B \triangle (A \triangle A) \\ &= (B \triangle A) \triangle A \\ &= (C \triangle A) \triangle A \\ &= C \triangle (A \triangle A) \\ &= C \triangle \emptyset \\ &= C, \end{aligned}$$

where we used the identity

$$(E \triangle F) \triangle G = E \triangle (F \triangle G),$$

to get from lines two to three and from lines four to five, we used (a) on line one and to get from line six to seven and we used (b) to get from line one to two and line five to line six.
5. Let $B = A \triangle \{ 1 \}$. There are two cases.
Suppose that $1 \notin A$. Then $B = A \cup \{ 1 \}$ has one more element than A. It follows that

$$|B| = |A| + 1.$$

Thus $|A|$ is even if and only if $|B|$ is odd.
Now suppose that $1 \in A$. Then $B = A \setminus \{ 1 \}$ has one fewer element than A. It follows that

$$|B| = |A| - 1 \qquad \text{so that} \qquad |A| = |B| + 1.$$

Thus $|B|$ is even if and only if $|A|$ is odd.
Taking the contrapositive of both implications, we conclude that $|A|$ is even if and only if $|B|$ is odd.
Either way $|A|$ is even if and only if $|B|$ is odd.
Challenge problems/Just for fun:
6. After quite a bit of trial and error, one realises the weights are 1, 3, 9 and 27.
One way to cut down on the trial and error is to realise that there are exactly forty different ways to use four stones on a scale. You could use 1, 2, 3 or all 4 stones. There is only one way to use one stone and so there are 4 ways to use one of the four stones. There are two ways to use two stones; put them all on side or put them either side. On the other hand, there are 6 ways to pick two stones from four stones. So

there are 12 ways to use two of the four stones. There are four ways to use three stones; either put them all on one side, or put one on one side and the other two on the other side. There are four ways to pick three stones from four and so there are $16 = 4 \times 4$ ways to use three of the four stones. Finally suppose we use all four stones. We could put all stones on one side, put one stone on one side, or divide the stones in two groups of two; there are then $1 + 4 + 3 = 8$ ways to use all four stones.

Putting all of this together we get $4 + 12 + 16 + 8 = 40$ ways to use all four stones. This means there can be no duplication. There can only be one way to weigh any weight between 1 and 40. So, for example, if one stone weighs one pound the difference between any other two stones is never one.

If there are n pieces, the weights are obviously

$$1, 3, 9, \ldots, 3^{n-1}.$$

This a geometric series with initial term 1 and ratio 3. The sum of the series is

$$w = \frac{1(1-3^n)}{1-3} = \frac{3^n - 1}{2}.$$

7. Jan puts the ring into a box, puts a padlock onto the box and sends it to Maria. The box has one padlock on it, Jan's. Maria puts her own padlock onto the same box and sends it back to Jan. The box now has two padlocks, one is Jan's and the other is Maria's. Jan removes his padlock and sends the box back to Maria. The box has one padlock on it, Maria's. Now Maria can unlock the box and take the ring.