MODEL ANSWERS TO THE SIXTH HOMEWORK

1. (a) This is true and so we prove it. Pick $x \in \mathbb{R}$. We have to prove

$$(\forall \epsilon > 0,\ |x| < \epsilon) \implies x = 0.$$

We prove the contrapositive, that is, we prove

$$x \neq 0 \implies (\exists \epsilon > 0,\ |x| \geq \epsilon)$$

If $x > 0$ then let $\epsilon = x/2$. We have

$$\begin{aligned} |x| &= x \\ &> x/2 \\ &= \epsilon. \end{aligned}$$

If $x < 0$ then let $\epsilon = -x/2$. We have

$$\begin{aligned} |x| &= -x \\ &> -x/2 \\ &= \epsilon. \end{aligned}$$

Either way, $|x| > \epsilon$. This proves the contrapositive and so we have proved

$$\forall x \in \mathbb{R},\ ((\forall \epsilon > 0,\ |x| < \epsilon) \implies x = 0)\,.$$

(b) This is false and so we disprove it, that is, we prove the negation

$$\exists x \in \mathbb{R},\ \exists \epsilon > 0,\ (|x| < \epsilon \wedge x \neq 0).$$

This is easy. Take $x = 1$ and $\epsilon = 2$. Then

$$|x| = 1 < 2 = \epsilon \qquad \text{and} \qquad x = 1 \neq 0.$$

Thus we have disproved

$$\forall x \in \mathbb{R},\ \forall \epsilon > 0,\ (|x| < \epsilon \implies x = 0).$$

2. (a) Pick $x \in X$. We have to show

$$\chi_{A \cap B}(x) = \chi_A(x)\chi_B(x).$$

There are two cases. If $\chi_{A\cap B}(x) = 1$ then $x \in A \cap B$. Therefore $x \in A$ and $x \in B$ and so $\chi_A(x) = 1$ and $\chi_B(x) = 1$. But then $\chi_A(x)\chi_B(x) = 1 \cdot 1 = 1$.

Otherwise $\chi_{A\cap B}(x) = 0$. In this case $x \notin A \cap B$. Therefore, either $x \notin A$ or $x \notin B$. Suppose $x \notin A$. Then $\chi_A(x) = 0$ and so

$$\begin{aligned} \chi_A(x)\chi_B(x) &= 0 \cdot 1 \\ &= 0. \end{aligned}$$

By symmetry, if $x \notin B$ then $\chi_A(x)\chi_B(x) = 0$.
In all three cases

$$\chi_{A\cap B}(x) = \chi_A(x)\chi_B(x).$$

and so

$$\chi_{A\cap B} = \chi_A\chi_B.$$

(b) Pick $x \in X$. We have to show

$$\chi_A(x) + \chi_{X\setminus A}(x) = \chi_X(x).$$

Note that $\chi_X(x) = 1$, independently of x, and so we have to show

$$\chi_A(x) + \chi_{X\setminus A}(x) = 1.$$

There are two cases. If $\chi_A(x) = 0$ then $x \notin A$. But then $x \in X \setminus A$ so that $\chi_{X\setminus A}(x) = 1$. It follows that the LHS is

$$\chi_A(x) + \chi_{X\setminus A}(x) = 0 + 1 = 1.$$

Otherwise $\chi_A(x) = 1$ and so $x \in A$. But then $x \notin X \setminus A$ so that $\chi_{X\setminus A}(x) = 0$. It follows that the LHS is

$$\chi_A(x) + \chi_{X\setminus A}(x) = 1 + 0 = 1.$$

Either way,

$$\chi_A + \chi_{X\setminus A} = \chi_X.$$

(c) We have to show

$$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A\chi_B(x).$$

There are four cases. Suppose that $x \in A$ but $x \notin B$. Then $x \in A \cup B$ and so the LHS is

$$\chi_{A\cup B}(x) = 1.$$

On the other hand, $\chi_A(x) = 1$ and $\chi_B(x) = 0$ so that the RHS

$$\chi_A(x) + \chi_B(x) - \chi_A\chi_B(x) = 1 + 0 - 1 \cdot 0 = 1.$$

We can prove the case $x \notin A$ and $x \in B$ using symmetry.
Now suppose that $x \in A$ and $x \in B$. Then $x \in A \cup B$ and so the LHS is

$$\chi_{A\cup B}(x) = 1.$$

On the other hand, $\chi_A(x) = 1$ and $\chi_B(x) = 1$ so that the RHS

$$\chi_A(x) + \chi_B(x) - \chi_A\chi_B(x) = 1 + 1 - 1 \cdot 1 = 1.$$

Finally suppose that $x \notin A$ and $x \notin B$. Then $x \notin A \cup B$ and so the LHS is

$$\chi_{A\cup B}(x) = 0.$$

On the other hand, $\chi_A(x) = 0$ and $\chi_B(x) = 0$ so that the RHS

$$\chi_A(x) + \chi_B(x) - \chi_A\chi_B(x) = 0 + 0 - 0 \cdot 0 = 0.$$

Since we have shown that

$$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A\chi_B(x).$$

in all four cases, we have proved that

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A\chi_B.$$

(d) We claim that

$$(X \setminus B) \cap A = A \setminus B.$$

It suffices to show a containment both ways. We first show that the LHS is a subset of the RHS. If $x \in (X \setminus B) \cap A$ then $x \in X \setminus B$ and $x \in A$, that is, $x \notin B$ and $x \in A$. Therefore $x \in A \setminus B$. Now we show RHS is a subset of the LHS. Suppose that $x \in A \setminus B$. Then $x \in A$ and $x \notin B$. As $x \in X$ we $x \in X \setminus B$. But then $x \in (X \setminus B) \cap A$. Thus we have equality.

Note that we proved in (b) that

$$\chi_B + \chi_{(X\setminus B)} = 1,$$

and so

$$\chi_{(X\setminus B)} = 1 - \chi_B.$$

We have

$$\begin{aligned}
\chi_{A\setminus B} &= \chi_{(X\setminus B)\cap A} \\
&= \chi_{(X\setminus B)}\chi_A \\
&= (1 - \chi_B)\chi_A \\
&= \chi_A - \chi_A\chi_B,
\end{aligned}$$

where we used the claim on the first line, part (a) to get from line one to line two and the observation above to get from line two to line three.

(e) We have

$$\begin{aligned}
\chi_{A\triangle B} &= \chi_{A\setminus B} + \chi_{B\setminus A} \\
&= \chi_A - \chi_A\chi_B + \chi_B - \chi_B\chi_A \\
&= \chi_A + \chi_B - 2\chi_A\chi_B,
\end{aligned}$$

where we use the definition of the symmetric difference on line one and part (d), twice, to get from line one to line two.

Note that $\chi_{A\triangle B}(x) = 1$ if and only if $\chi_{A\triangle B}(x)$ is odd.

On the other hand,

$$\chi_A(x) + \chi_B(x) - 2\chi_A\chi_B(x),$$

is odd, if and only if there is an integer k such that

$$\chi_A(x) + \chi_B(x) - 2\chi_A\chi_B(x) = 2k + 1.$$

Suppose that $l = \chi_A(x)\chi_B(x)$. Then

$$\chi_A(x) + \chi_B(x) = 2(k + l) + 1,$$

so that $\chi_A(x) + \chi_B(x)$ is odd, as $k + l$ is an integer. Therefore

$$(\chi_{A\triangle B}(x) = 1) \iff (\chi_A(x) + \chi_B(x) \text{ is odd}).$$

(f) Suppose that

$$\forall x \in X,\ \chi_A(x) \leq \chi_B(x).$$

We show that $A \subset B$.
Pick $a \in A$. Then, by definition of χ_A, we have $\chi_A(a) = 1$. As

$$1 = \chi_A(a) \leq \chi_B(a),$$

and the only possible values of $\chi_B(a)$ are 0 or 1, we must have $\chi_B(a) = 1$. But then by definition of χ_B, we have $a \in B$. It follows that $A \subset B$.
Now suppose that $A \subset B$. We show that

$$\forall x \in X,\ \chi_A(x) \leq \chi_B(x).$$

Pick $x \in X$. There are two cases.
If $\chi_B(x) = 0$ then, by definition of χ_B, we have $x \notin B$. As $A \subset B$ it follows that $x \notin A$. But then by definition of χ_A, $\chi_A(x) = 0$. In particular

$$\begin{aligned}\chi_A(x) &= 0 \\ &= \chi_B(x).\end{aligned}$$

Thus $\chi_A(x) \leq \chi_B(x)$.
Now suppose that $\chi_B(x) = 1$. As $\chi_A(x)$ is either 0 or 1, we have

$$\begin{aligned}\chi_A(x) &\leq 1 \\ &= \chi_B(x).\end{aligned}$$

Either way, $\chi_A(x) \leq \chi_B(x)$.
Thus

$$\forall x \in X,\ \chi_A(x) \leq \chi_B(x).$$

Note that $\chi_A = \chi_B$ if and only if $\chi_A \leq \chi_B$ and $\chi_B \leq \chi_A$. Similarly, $A = B$ if and only if $A \subset B$ and $B \subset A$. Therefore

$$(\chi_A = \chi_B) \iff (A = B).$$

3. We first check that Θ is injective. Suppose that A and $B \in \wp(X)$ and $\Theta(A) = \Theta(B)$. Then A and B are two subsets of X and $\chi_A = \chi_B$. 2 (f) implies that $A = B$. Thus Θ is injective.
Now we check that Θ is surjective. Suppose that $f \in \{0,1\}^X$. Then f is a function from X to $\{0,1\}$. Define a subset A of X as follows

$$A = \{ x \in X \mid f(x) = 1 \}.$$

We check that $\chi_A = f$. Both sides are functions from X to $\{0,1\}$. We check they have the same effect on $x \in X$.
There are two cases. If $f(x) = 1$ then $x \in A$, by definition of A, and so $\chi_A(x) = 1$. If $f(x) = 0$ then $x \notin A$, by definition of A, and so $\chi_A(x) = 0$. Either way, $\chi_A(x) = f(x)$ and so $f = \chi_A = \Theta(A)$. Thus Θ is surjective.
It follows that Θ is a bijection.
4. (a) Suppose that a_1 and $a_2 \in A$ and that $(g \circ f)(a_1) = (g \circ f)(a_2)$. Let $b_1 = f(a_1)$ and $b_2 = f(a_2)$. We have

$$\begin{aligned} g(b_1) &= g(f(a_1)) \\ &= (g \circ f)(a_1) \\ &= (g \circ f)(a_2) \\ &= g(f(a_2)) \\ &= g(b_2). \end{aligned}$$

As g is injective, it follows that $b_1 = b_2$. But then

$$\begin{aligned} f(a_1) &= b_1 \\ &= b_2 \\ &= f(a_2). \end{aligned}$$

As f is injective, it follows that $a_1 = a_2$. Therefore $f \circ g$ is injective.
(b) Suppose that $c \in C$. As g is surjective we can find $b \in B$ such that $g(b) = c$. As f is surjective, it follows that we can find $a \in A$ such that $f(a) = b$. We have

$$\begin{aligned} (g \circ f)(a) &= g(f(a)) \\ &= g(b) \\ &= c. \end{aligned}$$

Thus $g \circ f$ is surjective.
(c) This is immediate from (a) and (b).
Challenge problems/Just for fun:
5. We first show the easy direction. Suppose that we can marry off every boy. Then we can find an injective function from the set of boys

to the set of girls. If we have a subset of k boys then the image of this set under the function is a set of k girls. Thus every set of boys knows at least this many girls.

Now we turn to the hard direction. Suppose every subset of the boys knows at least as many girls. We find a way to marry off the boys.

Let m be the number of boys. We proceed by induction on m. If $m = 1$ there is one boy who knows at least one girl. Pick any one of the girls and marry them off.

Now suppose that know the result for any collection of at most m boys and that there are $m+1$ boys. Suppose that every collection of $k \leq m$ boys knows more girls. Pick any boy and marry them off to a girl. Then the remaining m boys have the property that any subset knows at least as many of the remaining girls. By induction we can marry off all the boys.

So we may assume that for some $k < m + 1$ there are k boys who know exactly k girls. By induction we can marry off all these boys. Consider a subset of the remaining $m + 1 - k$ boys. Suppose there are l boys who know fewer than l of the remaining girls. Consider this subset union the original k boys. There are $k + l$ boys and they will know at most $l - 1 + k$ girls, contrary to our assumption. As this is a contradiction, it must be the case that every subset of the $m + 1 - k$ remaining boys knows the at least as many girls, and so we can marry of these remaining boys by induction.