HOMEWORK 4, DUE WEDNESDAY FEBRUARY 8TH

1. Let R be a ring and let I be an ideal of R, not equal to the whole of R. Suppose that every element not in I is a unit. Prove that I is the unique maximal ideal in R.
2. Let $\phi\colon R \longrightarrow S$ be a ring homomorphism and suppose that J is a prime ideal of S.
(i) Prove that $I = \phi^{-1}(J)$ is a prime ideal of R.
(ii) Give an example of an ideal J that is maximal such that I is not maximal.
3. Prove that every prime element of an integral domain is irreducible.
4. (a) Show that the elements 2, 3 and $1 \pm \sqrt{-5}$ are irreducible elements of $\mathbb{Z}[\sqrt{-5}]$.
(b) Show that every element of R can be factored into irreducibles.
(c) Show that R is not a UFD.
Let R be a commutative ring. Our aim is to prove a very strong form of the Chinese Remainder Theorem. First we need some definitions. Let I and J be two ideals. The **sum** of I and J, denoted $I + J$, is the set consisting of all sums $i + j$, where $i \in I$ and $j \in J$. We say that I and J are **coprime** if $I + J = R$.
5. (a) Show that $I + J$ is an ideal of R.
(b) Show that I and J are coprime if and only if there is an $i \in I$ and a $j \in J$ such that $i + j = 1$.
(c) Show that if I and J are coprime then $IJ = I \cap J$.
Suppose that $I_1, I_2, \dots, I_k$ are ideals of R. We say these ideals are **pairwise coprime**, if for all $i \neq j$, I_i and I_j are coprime.
6. If $I_1, I_2, \dots, I_k$ are pairwise coprime, show that the product I of the ideals $I_1, I_2, \dots, I_k$ is equal to the intersection, that is

$$\prod_{i=1}^{k} I_i = \bigcap_{i=1}^{k} I_i.$$

(*Hint. Proceed by induction on* k).
Let R_i denote the quotient R/I_i. Define a map,

$$\phi\colon R \longrightarrow \bigoplus_{i=1}^{k} R_i,$$

by $\phi(a) = (a + I_1, a + I_2, \dots, a + I_k)$
7. (a) Show that ϕ is a ring homomorphism.

(b) See below.
(c) Show that ϕ is injective if and only if I, the intersection of the ideals $I_1, I_2, \ldots, I_k$, is equal to the zero ideal.

8. Deduce the Chinese Remainder Theorem, which states that if $I_1, I_2, \ldots, I_k$ are pairwise coprime and the product I is the zero ideal, then R is isomorphic to $\oplus_{i=1}^{k} R_i$. Show how to deduce the other versions of the Chinese Remainder Theorem, which are stated as exercises in the book.

Challenge Problems 7 (b) Show that ϕ is surjective if and only if the ideals $I_1, I_2, \ldots, I_k$ are pairwise coprime.

9. Let S be a commutative monoid, that is, a set together with a binary operation that is associative, commutative, and for which there is an identity, but not necessarily inverses. Treating this operation like multiplication in a ring, define what it means for S to have unique factorisation.

10. Let $v_1, v_2, \ldots, v_n$ be a sequence of elements of $\mathbb{Z}^2$. Let S be the semigroup that consists of all linear combinations of $v_1, v_2, \ldots, v_n$, with positive integral coefficients. Let the binary rule be ordinary addition. Determine which monoids have unique factorisation.

11. Show that there is a ring R, such that every element of the ring is a product of irreducibles, whilst at the same time the factorisation algorithm can fail.