## 14. Finitely Generated Modules over a PID

We want to give a complete classification of finitely generated modules over a PID. Recall that a finitely generated module is a quotient of $R^n$, a free module. Let $K$ be the kernel. Then $M$ is isomorphic to $R^n/K$, by the Isomorphism Theorem.

Now $K$ is a submodule of a Noetherian module; hence $K$ is finitely generated. Pick a finite set of generators of $K$ (it turns out that $K$ is also isomorphic to a free module. Thus $K$ is isomorphic to $R^m$, for some $m$, and in fact $m \leq n$).

As there is a map $R^m \longrightarrow K$, by composition we get an $R$-linear map

$$\phi\colon R^m \longrightarrow R^n.$$

Since $K$ is determined by $\phi$, $M$ is determined by $\phi$. The crucial piece of information is to determine $\phi$.

As this map is $R$-linear, just as in the case of vector spaces, everything is determined by the action of $\phi$ on the standard generators $f_1, f_2, \dots, f_m$. Suppose that we expand $\phi(f_i)$ as a linear combination of the standard generators $e_1, e_2, \dots, e_n$ of $R^n$.

$$\phi(f_i) = \sum_j a_{ij} e_j.$$

In this case we get a matrix

$$A = (a_{ij}) \in M_{n,m}(R).$$

The point is to choose different bases of $R^m$ and $R^n$ so that the representation of $\phi$ by $A$ is in a better form. Note the following:

**Lemma 14.1.** *Let $r_1, r_2, \dots, r_n$ be (respectively free) generators of $M$. Then so are $s_1, s_2, \dots, s_n$, where*

1. *we multiply one of the $r_i$ by an invertible element,*
2. *we switch the position of $r_i$ and $r_j$,*
3. *we replace $r_i$ by $r_i + ar_j$, where $a$ is any scalar.*

*Proof.* Easy. □

At the level of matrices, (14.1) informs us that we are free to perform any one of the elementary operations on matrices, namely multiplying a row (respectively column) by an invertible element, switching two rows (respectively columns) and taking a row and adding an arbitrary multiple of another row (respectively column).

**Proposition 14.2.** *Let $A$ be a matrix with entries in a Euclidean domain $R$.*

*Then, after a sequence of elementary row operations and column operations, we may put $A$ into the following form. The only non-zero entries are on the diagonal and each non-zero entry divides the next one in the list.*

*Proof.* The key point is to reduce to the case where one of the entries of $A$ is the gcd of the entries of $A$.

To this end, since the gcd of a sequence can be calculated by recursively taking the gcd of a pair, and by elementary row and column operations we can always make any two entries adjacent, we reduce to the case that $A$ is a $2 \times 1$ matrix,

$$\begin{pmatrix} a \\ b \end{pmatrix}.$$

Since we are working over a Euclidean domain (and not just a PID) we can calculate the gcd by using Euclid's algorithm. At each stage we may find $q$ and $r$ such that

$$b = qa + r \qquad \text{or} \qquad a = qb + r.$$

By symmetry we may assume we have the former case. Now either $r = 0$ in which case either $a$ is the gcd, and we are done, or by Euclid's algorithm it suffices to find the gcd of $a$ and $r$. But if we take the first row of $a$ and multiply by $q$ and subtract this from the second row then we get the matrix

$$\begin{pmatrix} a \\ r \end{pmatrix}.$$

Therefore, after finitely many elementary row and column operations we may assume that one entry of $A$ is the gcd.

Now by permuting the rows and columns, we may assume that $d$ is at the top left hand corner. As $d$ is the gcd, it divides every entry of $A$. By row and column reduction we reduce to the case that the only non-zero entry in the first column and row is the entry $d$ at the top left hand corner. Let $B$ be the matrix obtained by striking out the first row and column. Then every element of $B$ is divisible by $d$ and we are done by induction on $m$ and $n$. $\square$

**Remark 14.3.** *One can actually reduce any matrix over a PID into the same form. In this case one needs to pre- and post-multiply by invertible matrices with entries in $R$.*

As before we are reduced to the case

$$A = \begin{pmatrix} a \\ b \end{pmatrix}.$$

In the general case, as $R$ is a PID, note that we may find $x$ and $y$ such that

$$d = xa + yb.$$

Note that the gcd of $x$ and $y$ must be 1. Therefore we may find $u$ and $v$ such that

$$1 = ux + vy.$$

Let

$$B = \begin{pmatrix} x & y \\ -v & u \end{pmatrix}.$$

Note that the determinant of $B$ is

$$xu + yv = 1.$$

Thus $B$ is invertible, with inverse

$$\begin{pmatrix} u & -y \\ v & x \end{pmatrix}.$$

On the other hand,

$$BA = \begin{pmatrix} d \\ -va + ub \end{pmatrix}.$$

**Corollary 14.4.** *Let $M$ be a module over a PID $R$.*

*Then $M$ is isomorphic to $F \oplus T$, where $F$ is a free module and $T$ is isomorphic to, either*

*(1)*

$$R/\langle d_1 \rangle \oplus R/\langle d_2 \rangle \oplus \cdots \oplus R/\langle d_n \rangle,$$

*where $d_i$ divides $d_{i+1}$, or*

*(2)*

$$R/\langle p_1^{m_1} \rangle \oplus R/\langle p_2^{m_2} \rangle \oplus \cdots \oplus R/\langle p_n^{m_n} \rangle,$$

*where $p_i$ is a prime.*

*Proof.* By the Chinese Remainder Theorem it suffices to prove the first classification result. By assumption $M$ is isomorphic to a quotient of $R^n$ by an image of $R^m$. By (14.2) we may assume the corresponding matrix has the given simple form. Now note that the rows that contain only zeroes, correspond to the free part, and there is an obvious corrrespondence between the non-zero rows and the direct summands of the torsion part. □

One special case deserves attention:

**Corollary 14.5.** *Let $G$ be a finitely generated abelian group.*

*Then $G$ is isomorphic to $\mathbb{Z}^r \times T$, where $T$ is isomorphic to*

*(1)*

$$\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_n},$$

*where* $d_1, d_2, \ldots, d_n$ *are positive integers and* $d_i$ *divides* $d_{i+1}$, *or*

*(2)*

$$\mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_n^{m_n}}.$$

*where* $p_1, p_2, \ldots, p_n$ *are primes.*

Really the best way to illustrate the proof of these results, which are not hard, is to illustrate the methods by an example. Suppose we are given

$$\begin{pmatrix} 3 & 8 & 7 & 9 \\ 2 & 4 & 6 & 6 \\ 1 & 2 & 2 & 1 \end{pmatrix}$$

The gcd is 1. Thus we first switch the third and first rows.

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 6 & 6 \\ 3 & 8 & 7 & 9 \end{pmatrix}.$$

As we now have a 1 in the first row, we can now eliminate 2 and 3 from the first column, a la Gaussian elimination, to get

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 2 & 1 & 6 \end{pmatrix}.$$

Now eliminate the entries in the first row.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 2 & 1 & 6 \end{pmatrix}$$

Now we switch the second and third columns,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 1 & 2 & 6 \end{pmatrix}$$

and then the second and third rows,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 6 \\ 0 & 2 & 0 & 4 \end{pmatrix}$$

Now eliminate as before,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & -8 \end{pmatrix}$$

Now multiply the third row by $-1$ and eliminate the 8, to get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

This corrresponds to a $\mathbb{Z}$-linear map

$$\phi \colon \mathbb{Z}^4 \longrightarrow \mathbb{Z}^3.$$

It follows then that we have $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})/(\mathbb{Z} \oplus \mathbb{Z} \oplus 4\mathbb{Z}) \simeq \mathbb{Z}_4$. The free part is zero and the torsion part is $\mathbb{Z}_4$.

Suppose instead we have the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This represents a $\mathbb{Z}$-linear map

$$\mathbb{Z}^4 \longrightarrow \mathbb{Z}^5,$$

in the standard way. It follows then that we have

$$(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})/(\mathbb{Z} \oplus 3\mathbb{Z} \oplus 30\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{30} \simeq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_3 \times \mathbb{Z}_{30}.$$

The free part is $\mathbb{Z} \times \mathbb{Z}$ and the torsion part is $\mathbb{Z}_3 \times \mathbb{Z}_{30} \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.