15. Canonical Forms

Let $\phi\colon V \longrightarrow V$ be a linear map, where V is a finite dimensional vector space over the field F. Our goal is to understand ϕ. If we do the usual thing, which is to pick a basis for V, $v_1, v_2, \dots, v_n$ and expand ϕ in this basis, then we get a matrix $A = (a_{ij})$ and if we choose a different basis then we get a similar matrix, BAB^{-1}, where B is the matrix giving the change of basis.

The best one can hope for is that we can diagonalise A. But sometimes this does not work. If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then A has only one eigenvector and if a 2×2 matric is diagonalisable then there must be two independent eigenvectors.

Definition 15.1. *Let F be a field. We say that F is **algebraically closed** if every polynomial has a zero.*

Example 15.2. $\mathbb{R}$ *is not algebraically closed.*

Indeed $x^2 + 1$ does not have any real zeroes.

Theorem 15.3 (Fundamental Theorem of Algebra)**.** $\mathbb{C}$ *is algebraically closed.*

There are now two cases. If F is not algebraically closed then we don't try to be too clever. We just pick a vector v and look at its iterates. Eventually some iterate is a linear combination of the previous vectors. The matrix of this linear transformation looks like:

$$\begin{pmatrix} 0 & 0 & 0 & \dots & -a_0 \\ 1 & 0 & 0 & \dots & -a_1 \\ 0 & 1 & 0 & \dots & -a_2 \\ 0 & 0 & 1 & \dots & -a_3 \\ \vdots & \vdots & \vdots & & -a_{n-1} \end{pmatrix}$$

Here the last entries are $-a_0$, $-a_1$, $\dots$.

Definition 15.4. *Let $m(x)$ be the monic polynomial*

$$m(x) = x^n + \sum a_i x^i,$$

of degree n. The ***companion matrix*** *of $m(x)$ is the matrix*

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \dots & \ddots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}$$

Here the last row consists of the coefficients of $-m(x)$, not including the leading term.

If F is algebraically closed we can do much better.

Definition 15.5. *Let λ be a scalar. A* ***Jordan block*** *is a matrix of the form*

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & \dots \\ 0 & \lambda & 1 & 0 & \dots \\ 0 & 0 & \lambda & 1 & \dots \\ \vdots & \vdots & \vdots & \dots & \ddots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

The entries containing the ones above the main diagonal is called the super diagonal. Somewhat fancifully $J = A - \lambda I_n$ is the companion matrix associated to the zero polynomial.

Definition 15.6. *Let A be a matrix. We say that A is in* ***rational canonical form*** *if A is a block matrix, with zero matrices everywhere, except a bunch of square matrices containing the diagonal which are companion matrices of polynomials, $g_1, g_2, \dots, g_k$, where $g_i(x)$ divides $g_{i+1}(x)$.*

Definition 15.7. *Let A be a matrix. We say that A is in* ***Jordan canonical form*** *if A is a block matrix, with zero matrices everywhere, except a bunch of square matrices containing the diagonal which are Jordan blocks.*

Theorem 15.8 (Rational Canonical Form)**.** *Let $\phi\colon V \longrightarrow V$ be a linear map between finite dimensional vector spaces, over a field F.*

Then there is a basis $e_1, e_2, \dots, e_n$ such that the corresponding matrix is in rational canonical form. Equivalently every matrix A is similar to a matrix in rational canonical form. This decomposition is unique, if we order the blocks so that d_i divides d_{i+1}.

Proof. As $R = F[x]$ is a PID we may apply the classification of modules over a PID to conclude that V is isomorphic to the direct sum $R^r \oplus T$.

As R is an infinite dimensional vector space, it follows that $r = 0$. We can present T as

$$F[x]/\langle d_1(x)\rangle \oplus F[x]/\langle d_2(x)\rangle \oplus \cdots \oplus F[x]/\langle d_k(x)\rangle,$$

where d_i divides d_{i+1}. Now each direct summand corresponds to a block of our matrix. The action is given by multiplication by x. It follows that the action of ϕ preserves this decomposition, so that in block form we only get zero matrices off the main diagonal. So we might as well assume that there is only one summand (and then only one block).

Consider the action of ϕ with respect to the basis 1, x, x^2, ..., x^{n-1}, where n is the degree of $d(x)$. ϕ sends 1 to x, x to x^2 and so on. Now

$$x^k = \sum -a_i x^i,$$

where the entries $a_1, a_2, \dots, a_k$ are in the last column and

$$d(x) = x^n + \sum a_i x^i.$$

Taking transposes we get the companion matrix of $d(x)$. □

Theorem 15.9 (Jordan Canonical Form)**.** *Let $\phi\colon V \longrightarrow V$ be a linear map between finite dimensional vector spaces, over an algebraically closed field F.*

Then there is a basis $e_1, e_2, \dots, e_n$ such that the corresponding matrix is in Jordan canonical form. Equivalently every matrix A is similar to a matrix in Jordan canonical form.

Proof. V is isomorphic to

$$F[x]/\langle p_1^{m_1}(x)\rangle \oplus F[x]/\langle p_2^{m_2}(x)\rangle \oplus \cdots \oplus F[x]/\langle p_k^{m_k}(x)\rangle,$$

where each $p_i(x)$ is a prime (equivalently irreducible) polynomial. As before we might as well assume that there is only one summand (and then only one block).

Since F is algebraically closed, the only irreducible polynomials are in fact linear polynomials. Thus

$$p(x) = x - \lambda$$

for some $\lambda \in F$. Note that $m_1 = n$, so that V is isomorphic to

$$F[x]/\langle (x-\lambda)^n\rangle.$$

Consider the linear map $\psi = \phi - \lambda I$. For this action V is isomorphic to

$$F[y]/\langle y^n\rangle.$$

It is easy to see that if we put ψ into rational canonical form then the corresponding matrix for ϕ is a Jordan block. □