PRACTICE PROBLEMS FOR THE FIRST MIDTERM

1. Give the definition of:
   (i) a ring.
   (ii) a commutative ring.
   (iii) a zero divisor.
   (iv) an integral domain.
   (v) the characteristic of a ring.
   (vi) an ideal.
   (vii) the quotient of a ring by an ideal.
   (viii) a prime ideal.
   (ix) a maximal ideal.
   (x) the field of fractions of an integral domain.

2. Let R be a commutative ring and let a be an element of R. Prove that the set

$$\{ ra \mid r \in R \}$$

is an ideal of R.

3. Show that a commutative ring R is a field if and only if the only ideals in R are the zero-ideal $\{0\}$ and the whole ring R.

4. Let R be an integral domain and let I be an ideal. Show that R/I is a field if and only if I is a maximal ideal.

5. (i) Let R be an integral domain. If $ab = ac$, for $a \neq 0$, b, $c \in R$, then show that $b = c$.
(ii) Show that every finite integral domain is a field.

6. If

$$I_1 \subset I_2 \subset I_3 \subset \cdots \subset I_n \subset \cdots ,$$

is an ascending sequence of ideals in a ring R then the union I is an ideal.

7. Let R be a ring and let $S = M_{2,2}(R)$ be the ring of all 2×2 matrices with entries in R. If I is an ideal of S then show that there is an ideal J of R such that I consists of all 2×2 matrices with entries in J.

8. Let m and n be coprime integers. Prove that

$$\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \oplus \mathbb{Z}_n.$$

9. Construct a field with nine elements.