MIDTERM
MATH 200B, UCSD, WINTER 17

You have 50 minutes.

There are 5 problems, and the total number of points is 65. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Section instructor:______________________________

Section Time:______________________________

Problem	Points	Score
1	15	
2	15	
3	15	
4	10	
5	10	
6	10	
7	10	
Total	65	

1. (15pts) *Give the definition of a Noetherian module.*

An R-module M is Noetherian if every submodule is finitely generated.

(ii) *Give the definition of the tensor product.*

The tensor product of two R-modules M and N is an R-module $M \underset{R}{\otimes} N$ together with a bilinear map $u\colon M \times N \longrightarrow M \underset{R}{\otimes} N$ which is universal amongst all bilinear maps: given a bilinear map $f\colon M \times N \longrightarrow P$ there is a unique R-linear map $\phi\colon M \underset{R}{\otimes} N \longrightarrow P$ such that $\phi = f \circ u$.

(iii) *Give the definition of a Jordan block.*

A Jordan block is a square matrix with a scalar λ on the main diagonal, 1 on the super diagonal and zeroes everywhere else.

2. (15pts) (i) *Show that if*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

is a short exact sequence of R-modules then N is Noetherian if and only if M and P are Noetherian.

Suppose that N is Noetherian. Then M is Noetherian since it is isomorphic to a submodule of N and a submodule of M is therefore isomorphic to a submodule of N. If Q is a submodule of P then the inverse image L of Q is a submodule of N and generators for L map to generators of Q. Thus P is also Noetherian.
Suppose that M and P are Noetherian. If N_i is an increasing sequence of submodules of N then their images P_i in P and their inverse images M_i in M are increasing sequences of submodules. As M and P are Noetherian, both of these sequences stabilise. Since a submodule of N is determined by its image in P and its inverse in M it follows that the sequence N_i stabilises. But then N is Noetherian.

(ii) *Show that if R is Noetherian then R^n is Noetherian, for any positive integer n.*

Use the obvious short exact sequence

$$0 \longrightarrow R \longrightarrow R^n \longrightarrow R^{n-1} \longrightarrow 0$$

and induction on n.

(iii) Show that if R is a Noetherian ring then every finitely generated module is Noetherian.

If M is finitely generated then it is a quotient of R^n. R^n is Noetherian by (ii) and this implies M is Noetherian by (i).

3. (15pts) *Prove Hilbert's basis theorem.*

Let R be Noetherian. Hilbert's basis theorem states that $R[x]$ is Noetherian.
Suppose that I is an ideal in $R[x]$. We have to show that I is finitely generated.
Let $J \subset R$ be the subset of leading coefficients. J is clearly non-empty. Suppose that a and $b \in J$. Then we may find f and $g \in I$ with leading coefficients a and b. If f has degree m and g has degree n then multiplying through f by x^{n-m} or g by x^{m-n}, we may assume that $m = n$. In this case $f+g \in I$ has leading coefficient $a+b$. Thus $a+b \in J$ and J is closed under addition. As $rf \in I$ has leading coefficient ra, $ra \in J$ so that J is also closed under scalar multiplication. Thus J is an ideal in R.
As R is Noetherian, $J = \langle a_1, a_2, \ldots, a_k \rangle$, where $a_1, a_2, \ldots, a_k \in J \subset R$. Pick polynomials $f_1, f_2, \ldots, f_k \in I$ with leading coefficients $a_1, a_2, \ldots, a_k$. Let d_i be the degree of f_i and let d be the maximum degree. Let g be a polynomial of degree e contained in I. We first show that we can find polynomials g_0 and g_1 where $g = g_0 = g_1$, g_0 belongs to $I_0 = \langle f_1, f_2, \ldots, f_k \rangle$ and the degree of g_1 is less than d. We proceed by induction on e.
If the degree of g is less than e there is nothing to prove. If the degree of g is at least e then let a be the leading coefficient. As $a \in J$ we may find $r_1, r_2, \ldots, r_k$ such that $a = \sum r_i a_i$. Then

$$g - \sum r_i x^{e-d_i} f_i(x)$$

has degree less than e.
Let M be the set of all polynomials of degree at most e. Then M is a finitely generated submodule of $R[x]$. Thus M is Noetherian by 2(iii). As $I \cap M$ is a submodule of M it follows that we can find $h_1, h_2, \ldots, h_l$ such that every element of $I \cap M$ is an R-linear combination of $h_1, h_2, \ldots, h_l$.
Putting all of this together, every element $g \in I$ is a sum $g_0 + g_1$, where g_0 belongs to the ideal generated by $f_1, f_2, \ldots, f_k$ and g_1 is an R-linear combination of $h_1, h_2, \ldots, h_l$. Thus $f_1, f_2, \ldots, f_k$ and $h_1, h_2, \ldots, h_l$ generate I, so that I is finitely generated and $R[x]$ is Noetherian.

4. (10pts)
(i) *Let M and N and P be R-modules. Show that there is a natural isomorphism*

$$(M \oplus N) \underset{R}{\otimes} P \simeq M \underset{R}{\otimes} P \oplus N \underset{R}{\otimes} P.$$

To define a map left to right it suffices to define a linear map to either factor of the product; by definition of the tensor product it is then sufficient to define a bilinear map

$$M \oplus N \times P \longrightarrow M \underset{R}{\otimes} P$$

We just send (m, n, p) to $m \otimes p$. It is easy to this map is bilinear.
To define a map right to left it suffices to define a linear map from either factor of the direct sum; by definition of the tensor product it is then sufficient to define a bilinear map

$$M \times P \longrightarrow (M \oplus N) \underset{R}{\otimes} P$$

We just send (m, p) $(m, 0) \otimes p$. It is to see that this map is bilinear. It is then straightforward to check that these maps are inverses of each other.

(ii) *Identify*

$$\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n.$$

$$\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n \simeq \mathbb{Z}_d,$$

where d is the gcd of m and n.
Note that $1 \otimes 1$ generates the tensor product. We have

$$m(1 \otimes 1) = m \otimes 1 = 0 \otimes 1 = 0 \quad n(1 \otimes 1) \quad = 1 \otimes n = 1 \otimes 0 = 0.$$

As d is a linear combination of m and n it follows that $d(1 \otimes 1) = 0$, so that the tensor product is a cyclic group of order at most d.
Define a bilinear map

$$\mathbb{Z}_m \otimes \mathbb{Z}_n \longrightarrow \mathbb{Z}_d,$$

by sending (a, b) to ab, modulo d Note that if we change a by a multiple of m then the product changes by a multiple of m and if we change b by a multiple of n then the product changes by a multiple of n. Thus this is map is well-defined. It is easy to see it is bilinear. By definition of the tensor product we get a linear map

$$\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n \longrightarrow \mathbb{Z}.$$

As $1 \otimes 1$ is sent to 1 this map is surjective and the result follows.

5. (10pts) *Let A and B be two $n \times n$ matrices over a field F.*
(i) *If $n \leq 3$ show that A and B are similar if and only if they have the same minimal and characteristic polynomials.*

The minimal and characteristic polynomial of a matrix is an invariant of the underlying linear map. As two matrices are similar if and only if they represent the same linear map with respect to different bases, it is clear that similar matrices have the same minimal and characteristic polynomials.
Now suppose that $n \leq 3$ and A and B have the same minimal $m(x)$ and characteristic polynomials $c(x)$. We just have to show that A and B have the same rational canonical form. Let $d_1, d_2, \dots, d_k$ be the polynomials associated to the rational canonical form of A. It is enough to show these are determined by m and c. Note that $d_k = m$. Consider the degree of m. If it is three then $k = 1$ and $d_1 = m$. If it is one then $k = 3$ and $d_1 = d_2 = d_3 = m$. So we may assume that the degree of m is two. In this case $m = d_2$ and $k = 2$. As $c = d_1 d_2 = d_1 m$, $d_1 = c/m$ so that d_1 is determined. Thus A and B are similar.

(ii) *Show we can find A and B with the same minimal and characteristic polynomials and yet A and B are not similar, for $n \geq 4$.*

It is enough to do this for $n = 4$. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Both are in Jordan canonical form, so they are not similar (or observe that the first matrix has three independent eigenvectors with eigenvalue 0 and the second matrix only has two). Clearly $A \neq 0$ and $B \neq 0$ and yet $A^2 = B^2 = 0$. Thus both matrices have minimal polynomial x^2. As the characteristic polynomial has degree four and it has the same zeroes as the minimal polynomial, both A and B have characteristic polynomial x^4.

Bonus Challenge Problems

6. (10pts) *Let R be an integral domain with field of fractions Q. Show that*

$$Q/R \underset{R}{\otimes} Q/R = 0.$$

Consider

$$\frac{a}{b} \otimes \frac{c}{d},$$

where a, b, c and d belong to R.

We compute the product

$$d(\frac{a}{bd} \otimes \frac{c}{d}),$$

in two different ways. By linearity on the left we get

$$\frac{a}{b} \otimes \frac{c}{d}.$$

By linearity on the right we get

$$\frac{a}{bd} \otimes c = \frac{a}{bd} \otimes 0 = 0.$$

Thus

$$\frac{a}{b} \otimes \frac{c}{d} = 0.$$

As every element of the tensor product is a finite linear combination of these elements, it follows that the tensor product is zero.

7. (10pts) *Let M and N be R-modules over a ring R. Show that there is natural isomorphism*

$$\bigwedge^{d}(M \oplus N) \simeq \bigoplus_{i+j=d} \left(\bigwedge^{i} M \underset{R}{\otimes} \bigwedge^{j} N \right).$$