PRACTICE PROBLEMS FOR THE FIRST MIDTERM

1. Give the definition of:

(i) a module.
(ii) an R-linear map.
(iii) a finitely generated module.
(iv) a cyclic module.
(v) a free module.
(vi) a short exact sequence.
(vii) a Noetherian module.
(viii) a Noetherian ring.
(ix) a bilinear map.
(x) the tensor product.
(xi) a symmetric (alternating) bilinear map.
(xii) the symmetric (wedge) product.
(xiii) the determinant.
(xiv) a companion matrix.
(xv) a Jordan block.
(xvi) Rational canonical form.
(xvii) Jordan canonical form.

2. Show that $R[\![x]\!]$ is Noetherian.
3. Show that if M and N are two Noetherian R-modules over a ring R then $M \underset{R}{\otimes} N$ is Noetherian.
4. Show that

$$\mathbb{Q}/\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}/\mathbb{Z} = 0.$$

5. Define the rank of any finitely generated module over a PID.
6. Suppose that

$$M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

is a sequence of R-modules.
Show that

$$0 \longrightarrow \operatorname{Hom}_R(P, Q) \longrightarrow \operatorname{Hom}_R(N, Q) \longrightarrow \operatorname{Hom}_R(M, Q),$$

is left exact for all R-modules Q if and only if the first sequence is right exact.

7. Find the characteristic polynomials of

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 5 & 2 & -8 & -8 \\ -6 & -3 & 8 & 8 \\ -3 & -1 & 3 & 4 \\ 3 & 1 & -4 & -5 \end{pmatrix}.$$

Are A and B similar? Find their Jordan canonical form.