10. The Tango bundle

We want construct a simple bundle of rank $n-1$ over $\mathbb{P}^n$ for any n.

Lemma 10.1 (Serre)**.** *If E is a globally generated bundle of rank r on $\mathbb{P}^n$ and $r > n$ then there is an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus r-n} \longrightarrow E \longrightarrow F \longrightarrow 0,$$

where F is a bundle of rank n.

Proof. As E is globally generated, there is an exact sequence

$$0 \longrightarrow K \longrightarrow H^0(\mathbb{P}^n, E) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow 0.$$

The kernel K is the bundle

$$K = \{ (x, \sigma) \in \mathbb{P}^n \times H^0(\mathbb{P}^n, E) \mid \sigma(x) = 0 \}.$$

If we projectivise we get a morphism

$$f \colon \mathbb{P}(K) \longrightarrow \mathcal{O}_{\mathbb{P}^n}\mathbb{P}(\times H^0(\mathbb{P}^n, E))\mathbb{P}(H^0(\mathbb{P}^n, E)).,$$

where the second map is just projection onto the second factor. The fibres of f are just the zero sets of the corresponding function,

$$f^{-1}(\sigma) = \{ x \in \mathbb{P}^n \mid \sigma(x) = 0 \}.$$

Suppose that $h^0(\mathbb{P}^n, E) = N + 1$. If $m = n + N - r$ then m is the dimension of $\mathbb{P}(K)$. Thus the image of f has codimension at least $r - n$.

It follows that we can find a $(r - n - 1)$ dimensional linear space $\mathbb{P}(V)$ which avoids the image. The composition

$$V \otimes \mathbb{P}^n \longrightarrow E$$

defines a trivial subbundle of rank $r - n$. □

Corollary 10.2. *If E is a vector bundle of rank r on $\mathbb{P}^n$ and $r > n$ then there is an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(a)^{\oplus r-n} \longrightarrow E \longrightarrow F \longrightarrow 0,$$

where F is a bundle of rank n.

Lemma 10.3. *Let E be a globally generated bundle of rank r.*

Then E contains a trivial bundle of rank one if and only if $c_r(E) = 0$.

Proof. One direction is clear. If there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow F \longrightarrow 0$$

where F is a vector bundle then $c(E) = c(F)$ so that $c_r(E) = 0$.

For the other direction we may assume that $r \leq n$. We consider the same exact sequence as before,

$$0 \longrightarrow K \longrightarrow H^0(\mathbb{P}^n, E) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow 0.$$

As before we get we get a morphism

$$f \colon \mathbb{P}(K) \longrightarrow \mathbb{P}(H^0(\mathbb{P}^n, E)),$$

Suppose that f is surjective. Suppose that $h^0(\mathbb{P}^n, E) = N + 1$. If $m = n + N - r$ then m is the dimension on $\mathbb{P}(K)$ and so the general fibre $Z = f^{-1}(\sigma)$ has dimension $n - r$. But then a general section vanishes in codimension r, a contradiction. It follows that f is not surjective. If σ is not in the image then σ is nowhwere vanishing, so that σ defines a trivial sub line bundle. $\square$

We now turn to Tango's construction. We start with the Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow 0.$$

Suppose we take the $\wedge^{n-1}$th power of this sequence.

$$0 \longrightarrow \bigwedge^{n-2} T_{\mathbb{P}^n}(-1) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus \binom{n+1}{n-1}} \longrightarrow \bigwedge^{n-1} T_{\mathbb{P}^n}(-1) \longrightarrow 0.$$

Now for the last term we have

$$\begin{aligned} \bigwedge^{n-1} T_{\mathbb{P}^n}(-1) &\simeq \Omega^1_{\mathbb{P}^n}(1) \otimes \det T_{\mathbb{P}^n}(-1) \\ &\simeq \Omega^2_{\mathbb{P}^n}(2). \end{aligned}$$

Let

$$E = (\bigwedge^{n-2} T_{\mathbb{P}^n}(-1) \otimes \mathcal{O}_{\mathbb{P}^n}(-1))^*.$$

The dual sequence to the sequence above is

$$0 \longrightarrow T_{\mathbb{P}^n}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus \binom{n+1}{2}} \longrightarrow E \longrightarrow 0.$$

Thus E is globally generated of rank

$$r = \binom{n+1}{2} - n = \binom{n}{2}.$$

If $n \geq 3$ then $r \geq n$ and so there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus r-n} \longrightarrow E \longrightarrow E' \longrightarrow 0,$$

for some vector bundle E' of rank n. E' is globally generated as it is a quotient of a globally generated vector bundle. The top chern of E' is

$$\begin{aligned} c_n(E') &= c_n(E) \\ &= 0. \end{aligned}$$

Thus E' contains a trivial subbundle. Let F be the quotient,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow E' \longrightarrow F \longrightarrow 0.$$

Thus F is a vector bundle of rank $n-1$ with total chern class,

$$\begin{aligned} c(F) &= c(E') \\ &= c(E) \\ &= \frac{1}{c(T_{\mathbb{P}^n}(-2)} \\ &= \frac{1-2h}{(1-h)^{n+1}}. \end{aligned}$$

Finally we show that F is simple.

We will need the following three exact sequences:

$$0 \longrightarrow T_{\mathbb{P}^n}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus\binom{n+1}{2}} \longrightarrow E \longrightarrow 0.$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus r-n} \longrightarrow E \longrightarrow E' \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow E' \longrightarrow F \longrightarrow 0.$$

If we take the dual of the last exact sequence and tensor by F then we get

$$h^0(\mathbb{P}^n, F^* \otimes F) \leq h^0(\mathbb{P}^n, E'^* \otimes F).$$

Now take the dual of the second sequence and tensor by F to get

$$h^0(\mathbb{P}^n, E'^* \otimes F) \leq h^0(\mathbb{P}^n, E^* \otimes F).$$

So we are down to showing:

$$h^0(\mathbb{P}^n, E^* \otimes F) \leq 1.$$

Now tensor the second and third exact sequences with E^* to get

$$0 \longrightarrow E^* \longrightarrow E^* \otimes E' \longrightarrow E^* \otimes F \longrightarrow 0,$$

and

$$0 \longrightarrow E^{*\oplus r-n} \longrightarrow E^* \otimes E \longrightarrow E^* \otimes E' \longrightarrow 0,$$

Thus it remains to show that E is simple and $h^1(\mathbb{P}^n, E^*) = 0$.

Now

$$\begin{aligned} E^* &= \bigwedge^{n-2} T_{\mathbb{P}^n}(-1) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \\ &\simeq \bigwedge^{2} \Omega^1_{\mathbb{P}^n}(1) \otimes \det T_{\mathbb{P}^n}(-1) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \\ &\simeq \Omega^2_{\mathbb{P}^n}(2). \end{aligned}$$

Thus

$$h^0(\mathbb{P}^n, E^*) = h^1(\mathbb{P}^n, E^*) = 0.$$

If we tensor the first exact sequence with E^* then we get

$$0 \longrightarrow E^* \otimes T_{\mathbb{P}^n}(-2) \longrightarrow E^{*\oplus\binom{n+1}{2}} \longrightarrow E^* \otimes E \longrightarrow 0.$$

Taking the long exact sequence of cohomology gives

$$h^0(\mathbb{P}^n, E^* \otimes E) = h^1(\mathbb{P}^n, E^* \otimes T_{\mathbb{P}^n}(-2)).$$

Finally we tensor the Euler sequence with $E^*(-1)$ to get

$$0 \longrightarrow E^*(-2) \longrightarrow E^{*\oplus n+1} \longrightarrow E * \otimes T_{\mathbb{P}^n}(-2) \longrightarrow 0.$$

Taking the long exact sequence of cohomology, we are done, provided,

$$h^1(\mathbb{P}^n, E^*(-1)) = 0 \qquad \text{and} \qquad h^2(\mathbb{P}^n, E^*(-2)) = 0.$$

This follows from the Bott formula, since

$$E^* = \Omega^1_{\mathbb{P}^n}.$$

Putting all of this together we have

Theorem 10.4 (Tango)**.** *For every n there is a simple vector bundle of rank $(n-1)$ F on $\mathbb{P}^n$ with*

$$c(F) = \frac{1-2h}{(1-h)^{n+1}}.$$