12. Examples

We start with a useful:

Lemma 12.1. *Let E be a rank two vector bundle on $\mathbb{P}^n$ and let σ be a global section of E such that the local complete intersection $Z(\sigma) = Y$ has codimension two.*

Then E is decomposable if and only if Y is a complete intersection.

Proof. Suppose that E is decomposable. Then $E \sim \mathcal{O}_{\mathbb{P}^n}(a) \oplus \mathcal{O}_{\mathbb{P}^n}(b)$ for some a and b. Then $\sigma = (f, g)$ where f and g are homogeneous polynomials of degrees a and b. In this case $Y = Z(f) \cap Z(g)$.

Now suppose that Y is the intersection of the hypersurfaces W and V. Then there are homogeneous polynomials f and g of degrees a and b such that $W = Z(f)$ and $V = Z(g)$. Thus

$$f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a)) \qquad \text{and} \qquad g \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(b)).$$

The Koszul complex of the section

$$\sigma = (f, g) \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a)) \oplus H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(b))$$

gives the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(a+b)) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(a)) \oplus \mathcal{O}_{\mathbb{P}^n}(-(b)) \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

Thus we get a non-zero element of

$$\operatorname{Ext}^1_{\mathbb{P}^n}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-(a+b))) \simeq H^0(Y, \mathcal{O}_Y).$$

It is enough to show that the last group is one dimensional. In this case there is only one non-trivial extension of $\mathcal{I}_Y$ by $\mathcal{O}_{\mathbb{P}^n}(-(a+b))$, so that E is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-(a)) \oplus \mathcal{O}_{\mathbb{P}^n}(-(b))$.

Claim 12.2. $h^0(Y, \mathcal{O}_Y) = 1$ *for every complete intersection Y of codimension two in $\mathbb{P}^n$, $n \geq 3$.*

Proof of (12.2). From the long exact sequence of cohomology associated to

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(a+b)) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(a)) \oplus \mathcal{O}_{\mathbb{P}^n}(-(b)) \longrightarrow \mathcal{I}_Y \longrightarrow 0,$$

we see that

$$h^0(Y, \mathcal{I}_Y) = h^1(Y, \mathcal{I}_Y) = 0.$$

From the long exact sequence of cohomology associated to

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathbb{P}^n \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

we see that

$$h^0(Y, \mathcal{O}_Y) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 1 \qquad \square$$

$\square$

Lemma 12.3. *Let Y be a local complete intersection in $\mathbb{P}^n$. Then*

$$\omega_Y \simeq \omega_{\mathbb{P}^n} \otimes \det N_{Y/\mathbb{P}^n}.$$

In particular $\det N_{Y/\mathbb{P}^n}$ *is the restriction of a line bundle if and only if ω_Y is the restriction of a line bundle.*

Proof. The first result is adjunction. The second result follows, as

$$\omega_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1). \qquad \square$$

It is not hard to write down local complete intersection curves in $\mathbb{P}^3$, who canonical divisor is the restriction of a line bundle on $\mathbb{P}^3$ and yet the curve is not a complete intersection.

We start with an easier case.

Example 12.4. *Let Y be m reduced points $p_1, p_2, \dots, p_m$ in $\mathbb{P}^2$.*

As Y is zero dimensional it follows that every vector bundle on Y is trivial. Recall that we can apply Serre's result in $\mathbb{P}^2$ provided that $k < 3$. Thus we get vector bundles E of rank two on $\mathbb{P}^2$, which are extensions:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow E \longrightarrow \mathcal{I}_Y(k) \longrightarrow 0.$$

Note that E has Chern classes

$$c_1(E) = k$$
$$c_2(E) = m.$$

Suppose that $k = 1$. If L is a line that does not meet Y then the exact sequence above reduces to

$$0 \longrightarrow \mathcal{O}_L \longrightarrow E|_L \longrightarrow \mathcal{O}_L(1) \longrightarrow 0.$$

This sequence splits, as

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$

Thus the splitting type is $(1,0)$ and this is the generic splitting type.

If $k = 2$ then this argument breaks down as

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0.$$

In fact the generic splitting type is $(1,1)$.

If L is a line which meets one point x_i of S then $E|_L$ has a section $\sigma|_L$ which vanishes at x_i and nowhere else. In this case, the restriction of the short exact sequence above becomes

$$0 \longrightarrow \mathcal{O}_L(1) \longrightarrow E|_L \longrightarrow \mathcal{O}_L(1) \longrightarrow 0.$$

This splits, as

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0,$$

so that the splitting type of L is $(1,1)$. Thus the generic splitting type must be $(1,1)$.

In the case $k=2$ the set of jump lines is precisely the set of lines which meet two or more points of Y. The case $k=1$ is more subtle and in fact the set of jump lines is a curve of degree $m-1$ in the dual $\mathbb{P}^2$.

Example 12.5. *Let Y be a union of $d>1$ distinct lines $L_1, L_2, \dots, L_d$ in $\mathbb{P}^3$.*

As $L_i = H_i \cap G_i$ is the intersection of two hyperplanes it follows that

$$\begin{aligned} N_{Y/\mathbb{P}^3}|_{L_i} &= N_{L_i/\mathbb{P}^3} \\ &= N_{H_i/\mathbb{P}^3}|_{L_i} \oplus N_{G_i/\mathbb{P}^3}|_{L_i} \\ &= \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1). \end{aligned}$$

It follows that $N_{Y/\mathbb{P}^3}|_{L_i} = \mathcal{O}_1 2$. and so $N_{Y/\mathbb{P}^3} = \mathcal{O}_Y(2)$. By Serre's construction there is a rank two vector indecomposable bundle E, which fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow E \longrightarrow \mathcal{I}_Y(2) \longrightarrow 0,$$

with Chern classes

$$\begin{aligned} c_1(E) &= 2 \\ c_2(E) &= d. \end{aligned}$$

If we restrict E to a line that meets one of the L_i transversally, arguing as above, we see that the generic splitting type is $(1,1)$. If L is a line that meets two of the lines of Y transversally then the L is a jumping line, so that the locus of jumping lines contains a codimension two subvariety of $\mathbb{G}(1,3)$.

Example 12.6. *Let Y be the union of r elliptic curves C_i of degree d_i in $\mathbb{P}^3$.*

For any such curve, there is an exact sequence

$$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^3} \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow 0.$$

As the tangent bundle of an elliptic curve is trivial it follows that

$$\begin{aligned} \det N_{C/\mathbb{P}^3} &\simeq \det T_{\mathbb{P}^3}|_C \\ &= \mathcal{O}_C(4). \end{aligned}$$

Therefore

$$N_{Y/\mathbb{P}^3} = \mathcal{O}_Y(4).$$

There is then an associated rank two vector bundle E with

$$c_1(E) = 4$$
$$c_2(E) = \sum d_i.$$

As a special case, if we take two plane elliptic curves C_1 and C_2 sitting in different planes H_1 and H_2 then $d_1 = d_2 = 3$ and $F = E(-2)$ is a rank two vector bundle with

$$c_1(F) = 0$$
$$c_2(F) = 2.$$

Example 12.7. *Let Y be the union of r disjoint conics $D_1, D_2, \dots, D_r$ in $\mathbb{P}^3$.*

If $D \subset H$ is a conic sitting in a plane H then there is an exact sequence

$$0 \longrightarrow N_{D/H} \longrightarrow N_{D/\mathbb{P}^3} \longrightarrow N_{H/\mathbb{P}^3}|_D \longrightarrow 0.$$

It follows that

$$\begin{aligned} \det N_{D/\mathbb{P}^3} &\simeq N_{D/H} \otimes N_{H/\mathbb{P}^3}|_D \\ &\simeq \mathcal{O}_D(2) \otimes \mathcal{O}_D(1) \\ &= \mathcal{O}_D(3). \end{aligned}$$

and so

$$N_{Y/\mathbb{P}^3} = \mathcal{O}_Y(3).$$

The rank two vector bundle E associated to Y has Chern classes

$$c_1(E) = 3$$
$$c_2(E) = 2r.$$

The generic splitting type is $(2, 1)$.

Example 12.8. *Now suppose we pick a union Y of complete intersection curves Y_i in $\mathbb{P}^3$.*

Pick r pairs of natural numbers (a_i, b_i), with $a_i + b_i = p$ constant. Pick polynomials

$$f_i \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(a_i)) \qquad \text{and} \qquad g_i \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(b_i)).$$

Let $Y_i = Z(f_i) \cap Z(g_i)$. If we pick $f_1, f_2, \dots, f_r$ and $g_1, g_2, \dots, g_r$ appropriately, $Y_1, Y_2, \dots, Y_r$ are smooth pairwise disjoint curves. Let Y be their union.

The Koszul complex for Y_i is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-(a_i + b_i)) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-a_i) \oplus \mathcal{O}_{\mathbb{P}^3}(-b_i) \longrightarrow \mathcal{I}_{Y_i} \longrightarrow 0.$$

It follows that

$$\begin{aligned}\det N_{Y_i/\mathbb{P}^3} &\simeq \mathcal{O}_{\mathbb{P}^3}(-(a_i+b_i))|_{Y_i} \\ &\simeq \mathcal{O}_{\mathbb{P}^3}(-p)|_{Y_i}.\end{aligned}$$

Thus

$$\det N_{Y/\mathbb{P}^3} = \mathcal{O}_Y(p).$$

The associated rank 2 vector bundle E associated to Y has

$$\begin{aligned}c_1(E) &= p \\ c_2(E) &= \sum a_i b_i.\end{aligned}$$