14. Topological versus holomorphic classification II

We now turn to the problem of putting a holomorphic structure on every topological rank two vector bundle on $\mathbb{P}^3$. This means that given integers c_1 and c_2 such that $c_1c_2 \equiv 0 \mod 2$ then we have to find a holomorphic rank two vector bundle E with chern classes c_1 and c_2 and if c_1 is even, in addition we have to pick E so that

$$\alpha(E) \equiv h^0(\mathbb{P}^3, E(-k-2)) + h^1(\mathbb{P}^3, E(-k-2)) \mod 2$$

is both odd and even.

If we put

$$d(m) = \begin{cases} 1 & \text{if } m \geq 0 \text{ and } m \equiv 0 \mod 4 \\ 0 & \text{otherwise} \end{cases}$$

then note that

$$d(m) \equiv h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) \mod 2.$$

We are going to use the Serre construction.

Lemma 14.1. *Let $Y \subset \mathbb{P}^3$ be the codimension two zero locus of a section $\sigma \in H^0(\mathbb{P}^3, E)$ of a rank two vector bundle E such that $c_1(E) = 2k$.*

Then

$$\alpha(E) \equiv h^0(Y, \mathcal{O}_Y(k-2)) + d(k-2) \mod 2.$$

Proof. Consider the Koszul complex of the section σ

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2k) \longrightarrow E^* \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

By twisting with $\mathcal{O}_{\mathbb{P}^3}(k-2)$ we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-k-2) \longrightarrow E^*(k-2) \longrightarrow \mathcal{I}_Y(k-2) \longrightarrow 0.$$

Now

$$\begin{aligned} E^*(k-2) &\simeq E \otimes \det E^* \otimes \mathcal{O}_{\mathbb{P}^3}(-k-2) \\ &\simeq E(-k-2). \end{aligned}$$

Therefore the long exact sequence of cohomology associated to the short exact sequence gives

$$0 \longrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-k-2)) \longrightarrow H^0(\mathbb{P}^3, E(-k-2)) \longrightarrow H^0(\mathbb{P}^3, \mathcal{I}_Y(k-2)) \longrightarrow 0$$

and

$$0 \longrightarrow H^1(\mathbb{P}^3, E(-k-2)) \longrightarrow H^1(\mathbb{P}^3, \mathcal{I}_Y(k-2)) \longrightarrow 0.$$

Finally from the exact sequence

$$0 \longrightarrow \mathcal{I}_Y(k-2) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(k-2) \longrightarrow \mathcal{O}_Y(k-2) \longrightarrow 0,$$

one obtains

$$0 \to H^0(\mathbb{P}^3, \mathcal{I}_Y(k-2)) \longrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k-2)) \longrightarrow H^0(Y, \mathcal{O}_Y(k-2)) \longrightarrow H^1(\mathbb{P}^3, \mathcal{I}_Y(k-2)) \to 0.$$

Putting all of this together we get

$$\begin{aligned}\alpha(E) &= h^0(\mathbb{P}^3, E(-k-2)) + h^1(\mathbb{P}^3, E(-k-2)) \\ &= h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-k-2)) + h^0(\mathbb{P}^3, \mathcal{I}_Y(k-2)) + h^1(\mathbb{P}^3, \mathcal{I}_Y(k-2)) \\ &\equiv h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k-2)) + h^0(Y, \mathcal{O}_Y(k-2)) \\ &\equiv h^0(Y, \mathcal{O}_Y(k-2)) + d(k-2) \mod 2.\end{aligned}$$

□

Example 14.2. *Let Y be the intersection of two hypersurfaces V_a and V_b of degrees a and b, where $a \leq b$ and $a + b = 2k$ is even.*

The associated rank two vector bundle E splits as a direct sum $\mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b)$. We get

$$\begin{aligned}&h^0(\mathbb{P}^3, E(-k-2)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(\frac{b-a}{2} - 2)) \\ &h^1(\mathbb{P}^3, E(-k-2)) = 0\end{aligned}$$

and so

$$\alpha(E) = d(\frac{b-a}{2} - 2).$$

Lemma 14.3. *Let σ be a section of a rank two vector bundle E, where $c_1(E) = 2k$. If the zero scheme Y of σ is a disjoint union of r complete intersections $Y_i = V_{a_i} \cap V_{b_i}$, $i = 1, 2, 3, \ldots, r$ with $a_i \leq b_i$ and $a_i + b_i = 2k$ then*

$$\alpha(E) \equiv (r-1)d(k-2) + \sum_{i=1}^{r} d(k - a_i - 2) \mod 2.$$

Proof. We already know that

$$\begin{aligned}&\alpha(E) + d(k-2) \equiv h^0(Y, \mathcal{O}_Y(k-2)) \mod 2 \qquad \text{so that} \\ &\alpha(E) + d(k-2) \equiv \sum_{i=1}^{r} h^0(Y_i, \mathcal{O}_{Y_i}(k-2)) \mod 2.\end{aligned}$$

Let $E_i = \mathcal{O}_{\mathbb{P}^3}(a_i) \oplus \mathcal{O}_{\mathbb{P}^3}(b_i)$ be the rank two vector bundle associated to Y_i. Then

$$\alpha(E_i) + d(k-2) \equiv h^0(Y_i, \mathcal{O}_{Y_i}(k-2)) \mod 2,$$

so that

$$\alpha(E) + d(k-2) \equiv \sum_{i=1}^{r} (\alpha(E_i) + d(k-2)) \mod 2$$

that is,

$$\alpha(E) \equiv (r-1)d(k-2) + \sum_{i=1}^{r} \alpha(E_i) \mod 2.$$

However, we proved in the example that

$$\alpha(E_i) \equiv d(k-a_i-2) \mod 2. \qquad \square$$

Define a function

$$\Delta\colon \mathbb{Z}^2 \longrightarrow \mathbb{Z},$$

by the formula

$$\Delta(c_1, c_2) = c_1^2 - 4c_2.$$

Then define

$$\Delta(E) = \Delta(c_1(E), c_2(E)).$$

Note that

$$c_1(E(k)) = c_1(E) + 2k \qquad \text{and} \qquad c_2(E(k)) = c_2(E) + kc_1(E) + k^2,$$

and so

$$\begin{aligned} \Delta(E(k)) &= (c_1(E) + 2k)^2 - 4(c_2(E) + kc_1(E) + k^2)) \\ &= c_1(E)^2 - 4c_2(E) \\ &= \Delta(E). \end{aligned}$$

Further

$$\Delta(c_1, c_2) \equiv \begin{cases} 0 \mod 4 & \text{when } c_1 \text{ is even} \\ 1 \mod 4 & \text{when } c_1 \text{ is odd.} \end{cases}$$

Theorem 14.4 (Atiyah, Horrocks, Rees)**.** *Every topological rank two vector bundle on $\mathbb{P}^3$ has a holomorphic structure.*

Proof. Suppose that we have a pair of integers (c_1, c_2) such that $c_1 c_2 \equiv 0 \mod 2$.

We construct a holomorphic rank two vector bundle E with

$$\begin{aligned} c_1(E) &= c_1 \\ c_2(E) &= c_2. \end{aligned}$$

Let Y be the union of disjoint complete intersection curves of type (a_i, b_i) where $a_i + b_i = p$ is constant. If E is the associated rank two vector bundle then E has chern classes

$$\begin{aligned} c_1(E) &= p \\ c_2(E) &= \sum a_i b_i. \end{aligned}$$

We assume that $a_i \leq b_i$ so that $a_i \leq p/2$.

If we fix c_1 and c_2 then it is not hard to argue that we may find p, r and $a_1, a_2, \dots, a_r$ such

$$p^2 - 4\sum_{i=1}^{r} a_i b_i = \Delta(c_1, c_2).$$

Further we may arrange for p to have the same parity as c_1. Let

$$E' = E\left(\frac{c_1 - p}{2}\right).$$

Then

$$\begin{aligned} c_1(E') &= c_1(E) + c_1 - p = c_1 \\ c_2(E') &= c_2(E) + \frac{c_1 - p}{2} c_1(E) + \frac{(c_1 - p)^2}{4} = c_2. \end{aligned}$$

Now suppose that c_1 is even. We have to realise both values of the α-invariant. The α invariant of E' is

$$\begin{aligned} \alpha(E') &\equiv \alpha(E) \\ &\equiv (r-1)d(p/2 - 2) + \sum_{i=1}^{r} d(p/2 - a_i - 2) \mod 2. \end{aligned}$$

It is possible to show, by elementary but involved calculations in number theory, that we can both parities, as we vary p, r and $a_1, a_2, \dots, a_r$. □

We now consider the case of rank three.

Theorem 14.5 (Vogelaar)**.** *Every topological rank three bundle on $\mathbb{P}^3$ has a holomorphic structure.*

In particular

Corollary 14.6. *Every topological bundle on $\mathbb{P}^3$ has a holomorphic structure.*

We will need an extension of the Serre construction from codimension two to codimension three. The key point in the Serre construction is that since the line bundle

$$\det N_{Y/\mathbb{P}^n}(-k)$$

is trivial then there is extension

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow \mathcal{I}_Y(k) \longrightarrow 0$$

such that E is locally free.

For codimension three, the idea is to consider extensions

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow \mathcal{I}_Y(k) \longrightarrow 0$$

and to try to prove that E is locally free.

Theorem 14.7. *Let Y be a codimension three local complete intersection.*

If the bundle

$$\det N_{Y/\mathbb{P}^n}(-k)$$

is generated by two global sections t_1 and t_2 then there is a vector bundle E of rank three with two sections s_1 and s_2 that are linearly dependent over Y.