15. An argument of Ran

We recall a famous conjecture of Hartshorne:

Conjecture 15.1 (Hartshorne)**.** *Let $Y \subset \mathbb{P}^n$ be a smooth subvariety. If $2 \operatorname{codim} Y < \dim Y$ then Y is a complete intersection.*

The first thing to say is that (15.1) is sharp. For example, consider the Grassmannian $\mathbb{G}(1,4)$ of lines in $\mathbb{P}^4 = \mathbb{P}(V)$. Under the Plücker embedding this gets mapped into

$$Y \subset \mathbb{P}^9 = \mathbb{P}(\bigwedge^2 V).$$

Y has dimension 6 and codimension 3, but the ideal of Y is generated by quadrics.

It is interesting to specialise this conjecture to the case of codimension two. In this case the conjecture becomes interesting if Y has dimension at least five, that is, $n \geq 7$. By the Serre correspondence this translates to:

Conjecture 15.2. *Every vector bundle of rank two on $\mathbb{P}^7$ splits.*

There is very little evidence for (15.2). We present an argument of Ziv Ran which gives the best results.

Theorem 15.3. *Let $Y \subset \mathbb{P}^{m+2}$ be a locally complete intersection subvariety of codimension two. Let $N = N_{Y/\mathbb{P}^{m+2}}$ be the normal bundle. Let $d = c_2(N)$. Suppose that $\det N \simeq \mathcal{O}_Y(k)$.*

If

(1)

$$k \geq \frac{d}{m} + m,$$

or

(2) $d \leq m$

then Y is a complete intersection.

There is an argument due to Barth that if Y is smooth, the characteristic is zero and $m \geq 4$ that the condition $\det N \simeq \mathcal{O}_Y(d)$ is vacuous.

If $\det N \simeq \mathcal{O}_Y(k)$ then by adjunction we have

$$\omega_X = \mathcal{O}_Y(k - m - 3).$$

Therefore (i) means that ω_X is large relative to d and m.

The idea of the proof of (15.3) goes back to a simple observation of Severi. If Y is contained in a hyersurface X of degree u then every $(u+1)$-secant line to Y must be contained in X. Therefore there is no $(u+1)$-secant line to Y through a general point of projective space.

The idea is to try to reverse this argument. Severi showed that if Y is a surface in $\mathbb{P}^4$ and the 2-secants to Y don't span the whole of $\mathbb{P}^4$ then Y is contained in a quadric. We are going to generalise this argument. If the $(m+1)$-secants to Y don't pass through a general point of $\mathbb{P}^{m+2}$ then Y is a contained in a hyersurface of degree at most m.

Let E be the rank two vector bundle associated to Y. We have the Koszul complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{m+2}} \longrightarrow E \longrightarrow \mathcal{I}_Y(k) \longrightarrow 0$$

We have

$$\begin{aligned} c_1(E) &= k \\ c_2(E) &= d. \end{aligned}$$

Define a function

$$e\colon \mathbb{Z} \longrightarrow \mathbb{Z}$$

by the rule

$$\begin{aligned} e(t) &= c_2(E(-t)) \\ &= c_2(E) - c_1(E)t + t^2 \\ &= d - t(k-t). \end{aligned}$$

Consider the incidence correspondence for the Grassmannian $\mathbb{G}(1, m+2))$ of lines in $\mathbb{P}^{m+2}$,

$$I = \{\, (p, L) \in \mathbb{P}^{m+2} \times \mathbb{G}(1, m+2) \,|\, p \in L \,\} \subset \mathbb{P}^{m+2} \times \mathbb{G}(1, m+2).$$

There are two natural projections

$$\begin{array}{ccc} I & \xrightarrow{\;g\;} & \mathbb{G}(1, m+2). \\ {\scriptstyle f}\big\downarrow & & \\ \mathbb{P}^{m+2} & & \end{array}$$

Let p be a general point of $\mathbb{P}^{m+2}$ and let

$$\Sigma_u = \Sigma_{u,p} = \{\, L \in \mathbb{G}(1, m+2) \,|\, \text{the length of } L \cap Y \text{ is at least } k \,\}.$$

We think of Σ_u as the set of u-secant lines to Y. If the intersection $L \cap Y$ is reduced then L is u-secant. If $u = 2$ and $L \cap Y$ is not reduced then L is tangent to Y at the point $L \cap Y$ and of course tangent lines are limits of secant lines.

There are strong analogies between the splitting type of a line and the number of times it is secant. The fact that Σ_u is non-empty is akin to the existence of jumping lines.

Proposition 15.4. *If none of the integers from* 0 *to* u *are roots of the polynomial* $e(t)$ *and* $u \leq m$ *then* Σ_{u+1} *is non-empty.*

Proof. We will prove the stronger statement that $\dim \Sigma_i = 2(m+1)-i$, for $i \leq u+1$.

Note that Σ_0 has dimension $2(m+1)$. So by induction it suffices to prove that $\Sigma_{u+1} \subset \Sigma_u$ is a divisor. For this it suffices to show that if that $C \subset \Sigma_u$ is an irreducible curve and C does not intersect Σ_{u+1} then $e(u) = 0$.

Our hypotheses imply that the length of $L \cap Y$ is equal to u for all $L \in C$. Let $\tilde{C} \longrightarrow C$ be the normalisation of C and let $\gamma\colon S \longrightarrow \tilde{C}$ be the pullback of the $\mathbb{P}^1$ bundle $g\colon I \longrightarrow \mathbb{G}(1, m+2)$. Let $\phi\colon S \longrightarrow \mathbb{P}^{m+2}$ be the natural map. Let $D = \pi^{-1}(Y)$, where we pullback Y as a scheme.

Consider the map $D \longrightarrow \tilde{C}$. By assumption the length of a fibre is equal to u, a constant. Therefore D is flat over $\tilde{C}$, hence Cohen-Macaulay. But then D is a Cartier divisor in S.

Let F be fibre of π and let Z be the section of π contracted down by ϕ to p. Let H be the pullback of a hyperplane. Then there is a numerical equivalence

$$D \equiv uH + lF,$$

for some l. But as $D \cdot Z = 0$, it follows that $l = 0$, so that

$$D \equiv uH.$$

As the map of sheaves

$$\mathcal{O}_S \longrightarrow \phi^* E.$$

vanishes on D, there is a short sequence

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \phi^* E \longrightarrow Q \longrightarrow 0,$$

where Q is a line bundle. If we use this to compute chern classes then we get

$$\begin{aligned} c_1(Q) &= c_1(\phi^* E) - c_1(\mathcal{O}_S(D)) \\ &= (k-u)H \end{aligned}$$

and so

$$\begin{aligned} c_2(\phi^* E) &= c_1(\mathcal{O}_S(D)) c_1(Q) \\ &= (k-u)u. \end{aligned}$$

But then $d = u(k-u)$, that is, $e(u) = 0$. $\square$

Corollary 15.5. *If none of the integers from* 0 *to* u *are roots of the polynomial* $e(t)$ *and* $u \leq m$ *then* Y *is not contained in a hypersurface of degree* k*.*

Proof. (15.4) implies that Σ_{u+1} is non-empty. But then Y is not contained in a hypersurface of degree k. $\square$

We now prove (15.3).