2. The Tangent bundle and projective bundle

Let us give the first non-trivial example of a vector bundle on $\mathbb{P}^n$. Recall that given any smooth projective variety one can construct the tangent bundle. Geometrically a tangent vector at $x \in X$ is an equivalence class of paths,

$$\gamma: (-\epsilon, \epsilon) \rightarrow X$$

such that $\gamma(0) = x$. Two paths are considered equivalent if they have the same first derivative (one can make sense of this in a way which is not circular). The set of all tangent vectors based at $x$ is a vector space of dimension $n$, $T_xX$. The tangent $TX$ bundle is the set of all tangent vectors. There is an obvious projection down to $X$, $\pi: TX \rightarrow X$. The fibre over a point is the tangent bundle. Since $X$ is locally isomorphic to an open subset of $\mathbb{R}^n$ and the tangent bundle of $\mathbb{R}^n$ is a product, it is clear that the tangent bundle is locally a product. The transition functions are given by the Jacobian of the coordinate change. Thus the tangent bundle is a bundle.

The algebraic approach to the construction of the tangent bundle proceeds from a different direction. If $X$ is a variety and $x \in X$ is a point, with local ring $\mathcal{O}_{X,x}$, then the Zariski tangent space is the dual of

$$T_xX = \frac{m^*}{m^2},$$

where $m$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. In terms of schemes, one can look at the set of maps

$$\text{Hom}(\text{Spec} \frac{\mathbb{C}[\epsilon]}{\langle \epsilon^2 \rangle}, X),$$

where the unique point of

$$\text{Spec} \frac{\mathbb{C}[\epsilon]}{\langle \epsilon^2 \rangle}$$

is sent to $x$. This is again the dual of the Zariski tangent space. If $X$ is affine, one can construct the sheaf of differentials. One can globalise this construction in a somewhat bizarre way. Let

$$\Delta: X \rightarrow X \times X,$$

be the diagonal morphism. Let $\mathcal{I}$ be the ideal sheaf of the diagonal sitting inside the product. Then

$$\frac{\mathcal{I}}{\mathcal{I}^2}.$$
is naturally supported on the diagonal, a copy of $X$. In fact if $X$ is smooth this sheaf is locally free and the pullback to $X$ is the cotangent sheaf $\Omega^1_X$, the sheaf of sections of the cotangent bundle.

There is a standard way to construct the tangent and cotangent bundles on projective space. Recall that as a manifold, $\mathbb{P}^n$ is the set of lines in an $(n + 1)$ dimensional vector space $V \simeq \mathbb{C}^{n+1}$. Almost by definition there is a universal sublinebundle

$$S \subset V \times \mathbb{P}^n$$

There is a quotient vector bundle $Q$, so that we get an exact sequence of vector bundles

$$0 \rightarrow S \rightarrow \mathbb{P}^n \times V \rightarrow Q \rightarrow 0.$$

A morphism to projective space is given by a line bundle and a choice of $n+1$ sections which don’t vanish simultaneously (the universal property of projective space). In this case the line bundle is the pullback of $S^*$. So a morphism of

$$\text{Spec} \frac{\mathbb{C}[\epsilon]}{(\epsilon^2)}$$

is given by a choice of deformation of the line bundle $S$. But if you deform in the direction of $S$ nothing happens. So the Zariski tangent space to $\mathbb{P}^n$ is

$$\text{Hom}(S, Q) \simeq Q \otimes S^*.$$

The exact sequence

$$0 \rightarrow S \rightarrow \mathbb{P}^n \times V \rightarrow Q \rightarrow 0.$$

is the **Euler sequence**. At the level of sheaves we have

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1} \rightarrow T_X(-1) \rightarrow 0.$$

If we tensor by $\mathcal{O}_{\mathbb{P}^n}$ we get

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \rightarrow T_X \rightarrow 0.$$

This second map sends $(l_0, l_1, \ldots, l_n)$ to

$$\sum l_i \frac{\partial}{\partial x_i}.$$

The kernel is $(x_0, x_1, \ldots, x_n)$ as

$$\sum x_i \frac{\partial}{\partial x_i}$$

is radial.

If $E$ is a vector bundle of rank $r$ we can associate a projective bundle over $X$.  

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Definition 2.1. A **projective bundle** \( Y \) on a complex projective variety \( X \) is a projective variety together with a holomorphic map \( \pi: Y \to X \) and an open cover \( \{ U_\alpha \} \) of \( X \) such that

\[
Y|_{U_\alpha} = \pi^{-1}(U_\alpha)
\]

is isomorphic to the product \( U_\alpha \times \mathbb{P}^r \) over \( U_\alpha \),

\[
\begin{array}{ccc}
E|_{U_\alpha} & \to & U_\alpha \times \mathbb{P}^r \\
\downarrow & & \downarrow \\
U_\alpha,
\end{array}
\]

such that on the overlap

\[
U_{\alpha\beta} = U_\alpha \cap U_\beta,
\]

the transition functions are linear functions on \( \mathbb{P}^r \).

If \( E \) is a vector bundle then one can construct the associated projective bundle, \( \mathbb{P}(E) \). By definition of \( E \), we can find an open cover \( \{ U_\alpha \} \) of \( X \) such that \( E_\alpha \cong X \times \mathbb{C}^r \). For the associated projective bundle, \( Y = \mathbb{P}(E) \), let \( Y_\alpha \cong X \times \mathbb{P}^{r-1} \). As the transition functions of \( E \) are given by linear functions then so are the transition functions for \( Y \). Thus \( Y \) is a projective bundle.

One can also make this construction algebraically. \( Y \) comes with a locally free sheaf \( \mathcal{O}_Y(1) \) of rank one. Fibre by fibre it restricts to the sheaf \( \mathcal{O}_{\mathbb{P}^{r-1}}(1) \). Note that two vector bundles \( E_1 \) and \( E_2 \) give rise to isomorphic projective bundles \( Y_1 \) and \( Y_2 \) if and only if there is a line bundle \( L \) such that \( E_1 = L \otimes E_2 \). In fact one direction is clear, since tensoring by a line bundle won’t change the fibres of the projective bundle, the transition functions of \( Y_1 \) and \( Y_2 \) are the same. Thus \( Y_1 \) and \( Y_2 \) are isomorphic. Note however that the tautological rank one sheaves differ,

\[
\mathcal{O}_{Y_2}(1) = \mathcal{O}_{Y_1}(1) \otimes \pi^* L.
\]

In general, a projective bundle \( Y \) over \( X \) won’t come from a vector bundle. It will come from a vector bundle if the open cover trivialising \( Y \) over \( X \) are Zariski open subsets and \( X \) is smooth. In this case, there is a divisor \( D \) on \( Y \), which restricts to the general fibre of \( \pi \) as a hyperplane. Just take the closure of the inverse image of \( U_\alpha \times H \), where \( H \) is a hyperplane in \( \mathbb{P}^{r-1} \). Consider the associated rank one locally free sheaf \( \mathcal{O}_Y(D) \). Standard results imply that

\[
\mathcal{E} = \pi_*(\mathcal{O}_Y(D)),
\]

is a locally free sheaf of rank \( r \).
However there are examples of projective bundles which are trivial in the Euclidean topology which don’t come from vector bundles. Consider the exact sequence of algebraic groups,
\[ 0 \longrightarrow \mathbb{C}^* \longrightarrow \text{GL}(r) \longrightarrow \text{PGL}(r) \longrightarrow 0. \]
One can sheafify this sequence to get
\[ 0 \longrightarrow \mathcal{O}_X^* \longrightarrow \text{GL}(r) \longrightarrow \text{PGL}(r) \longrightarrow 0. \]
Taking the long exact sequence of cohomology we get
\[ H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X, \text{GL}(r)) \longrightarrow H^1(X, \text{PGL}(r)) \longrightarrow H^2(X, \mathcal{O}_X^*). \]
Note that it does make sense to take cohomology of a sheaf of non-abelian groups. Note however that higher cohomology is no longer a group, just a pointed set. The cohomology set
\[ H^1(X, \text{GL}(r)) \]
classifies vector bundles of rank \( r \). The cohomology set
\[ H^1(X, \text{PGL}(r)) \]
classifies projective bundles of rank \( r - 1 \). The map between them is the natural map which assigns to a vector bundle the associated projective bundle. The kernel of this map is
\[ H^1(X, \mathcal{O}_X^*) \]
which as we have already seen classifies line bundles on \( X \). However the last map
\[ H^1(X, \text{PGL}(r)) \rightarrow H^2(X, \mathcal{O}_X^*). \]
is not always zero. The image is the Brauer group; it classifies projective bundles over \( X \) which are not Zariski trivial.

There is a fun example of a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^2 \). Let
\[ Y = \{ (x, L) \mid x \in L \} \subset \mathbb{P}^2 \times \mathbb{P}^2 \]
be the incidence correspondence between points and lines on \( \mathbb{P}^2 \). In coordinates \([x : y : z]\) on the first \( \mathbb{P}^2 \) and \([a : b : c]\) on the second \( \mathbb{P}^2 \), \( Y \) is given by the bihomogeneous equation
\[ ax + by + cz = 0. \]
Consider projection \( \pi \) of \( Y \) down to the first \( \mathbb{P}^2 \). The fibre over a point \([x : y : z]\) is the set of all lines this point. Fix the point \( p = [0 : 0 : 1] \). The set of lines through \( p \) is given by \( c = 0 \), so that we get the line \([a : b : 0] \subset \mathbb{P}^2 \). Thus the fibres of \( \pi \) are copies of \( \mathbb{P}^1 \). Now suppose we look at the affine open subset \( z \neq 0 \) of \( \mathbb{P}^2 \).

We can use point-slope to see that \( Y \) is trivial over \( U = \mathbb{A}^2 = (z \neq 0) \). More geometrically, a line through the point \([x : y : z]\) will meet the
line $L_2$, given by $z = 0$, at a unique point. Since a line is specified by two points, it is easy to see that $Y$ is isomorphic to $U \times L_2 \simeq U \times \mathbb{P}^1$. 