2. The Tangent bundle and projective bundle

Let us give the first non-trivial example of a vector bundle on $\mathbb{P}^n$. Recall that given any smooth projective variety one can construct the tangent bundle. Geometrically a tangent vector at $x \in X$ is an equivalence class of paths,

$$\gamma\colon (-\epsilon, \epsilon) \longrightarrow X$$

such that $\gamma(0) = x$. Two paths are considered equivalent if they have the same first derivative (one can make sense of this in a way which is not circular). The set of all tangent vectors based at x is a vector space of dimension n, $T_x X$. The tangent TX bundle is the set of all tangent vectors. There is an obvious projection down to X, $\pi\colon TX \longrightarrow X$. The fibre over a point is the tangent bundle. Since X is locally isomorphic to an open subset of $\mathbb{R}^n$ and the tangent bundle of $\mathbb{R}^n$ is a product, it is clear that the tangent bundle is locally a product. The transition functions are given by the Jacobian of the coordinate change. Thus the tangent bundle is a bundle.

The algebraic approach to the construction of the tangent bundle proceeds from a different direction. If X is a variety and $x \in X$ is a point, with local ring $\mathcal{O}_{X,x}$, then the Zariski tangent space is the dual of

$$T_x X = \frac{\mathfrak{m}}{\mathfrak{m}^2}^*,$$

where $\mathfrak{m}$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. In terms of schemes, one can look at the set of maps

$$\operatorname{Hom}(\operatorname{Spec} \frac{\mathbb{C}[\epsilon]}{\langle \epsilon^2 \rangle}, X),$$

where the unique point of

$$\operatorname{Spec} \frac{\mathbb{C}[\epsilon]}{\langle \epsilon^2 \rangle}$$

is sent to x. This is again the dual of the Zariski tangent space. If X is affine, one can construct the sheaf of differentials. One can globalise this construction in a somewhat bizarre way. Let

$$\Delta\colon X \longrightarrow X \times X,$$

be the diagonal morphism. Let $\mathcal{I}$ be the ideal sheaf of the diagonal sitting inside the product. Then

$$\frac{\mathcal{I}}{\mathcal{I}^2},$$

is naturally supported on the diagonal, a copy of X. In fact if X is smooth this sheaf is locally free and the pullback to X is the cotangent sheaf Ω^1_X, the sheaf of sections of the cotangent bundle.

There is a standard way to construct the tangent and cotangent bundles on projective space. Recall that as a manifold, $\mathbb{P}^n$ is the set of lines in an $(n+1)$ dimensional vector space $V \simeq \mathbb{C}^{n+1}$. Almost by definition there is a universal sublinebundle

$$S \subset V \times \mathbb{P}^n$$

There is a quotient vector bundle Q, so that we get an exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow \mathbb{P}^n \times V \longrightarrow Q \longrightarrow 0.$$

A morphism to projective space is given by a line bundle and a choice of $n+1$ sections which don't vanish simultaneously (the universal property of projective space). In this case the line bundle is the pullback of S^*. So a morphism of

$$\operatorname{Spec} \frac{\mathbb{C}[\epsilon]}{\langle \epsilon^2 \rangle}$$

is given by a choice of deformation of the line bundle S. But if you deform in the direction of S nothing happens. So the Zariski tangent space to $\mathbb{P}^n$ is

$$\operatorname{Hom}(S, Q) \simeq Q \otimes S^*.$$

The exact sequence

$$0 \longrightarrow S \longrightarrow \mathbb{P}^n \times V \longrightarrow Q \longrightarrow 0.$$

is the **Euler sequence**. At the level of sheaves we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow T_X(-1) \longrightarrow 0.$$

If we tensor by $\mathcal{O}_{\mathbb{P}^n}$ we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \longrightarrow T_X \longrightarrow 0.$$

This second map sends $(l_0, l_1, \dots, l_n)$ to

$$\sum l_i \frac{\partial}{\partial x_i}.$$

The kernel is $(x_0, x_1, \dots, x_n)$ as

$$\sum x_i \frac{\partial}{\partial x_i}$$

is radial.

If E is a vector bundle of rank r we can associate a projective bundle over X.

Definition 2.1. *A **projective bundle** Y on a complex projective variety X is a projective variety together with a holomorphic map $\pi\colon Y \longrightarrow X$ and an open cover $\{ U_\alpha \}$ of X such that*

$$Y|_{U_\alpha} = \pi^{-1}(U_\alpha)$$

is isomorphic to the product $U_\alpha \times \mathbb{P}^r$ over U_α,

$$\begin{array}{ccc} E|_{U_\alpha} & \longrightarrow & U_\alpha \times \mathbb{P}^r \\ & \searrow \quad \swarrow & \\ & U_\alpha, & \end{array}$$

such that on the overlap

$$U_{\alpha\beta} = U_\alpha \cap U_\beta,$$

the transition functions are linear functions on $\mathbb{P}^r$.

If E is a vector bundle then one can construct the associated projective bundle, $\mathbb{P}(E)$. By definition of E, we can find an open cover $\{ U_\alpha \}$ of X such that $E_\alpha \simeq X \times \mathbb{C}^r$. For the associated projective bundle, $Y = \mathbb{P}(E)$, let $Y_\alpha \simeq X \times \mathbb{P}^{r-1}$. As the transition functions of E are given by linear functions then so are the transition functions for Y. Thus Y is a projective bundle.

One can also make this construction algebraically. Y comes with a locally free sheaf $\mathcal{O}_Y(1)$ of rank one. Fibre by fibre it restricts to the sheaf $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$. Note that two vector bundles E_1 and E_2 give rise to isomorphic projective bundles Y_1 and Y_2 if and only if there is a line bundle L such that $E_1 = L_2 \underset{\mathcal{O}_X}{\otimes} E_2$. In fact one direction is clear, since tensoring by a line bundle won't change the fibres of the projective bundle, the transition functions of Y_1 and Y_2 are the same. Thus Y_1 and Y_2 are isomorphic. Note however that the tautological rank one sheaves differ,

$$\mathcal{O}_{Y_2}(1) = \mathcal{O}_{Y_1}(1) \underset{\mathcal{O}_{Y_1}}{\otimes} \pi^* L.$$

In general, a projective bundle Y over X won't come from a vector bundle. It will come from a vector bundle if the open cover trivialising Y over X are Zariski open subsets and X is smooth. In this case, there is a divisor D on Y, which restricts to the general fibre of π as a hyperplane. Just take the closure of the inverse image of $U_\alpha \times H$, where H is a hyperplane in $\mathbb{P}^{r-1}$. Consider the associated rank one locally free sheaf $\mathcal{O}_Y(D)$. Standard results imply that

$$\mathcal{E} = \pi_*(\mathcal{O}_Y(D)),$$

is a locally free sheaf of rank r.

However there are examples of projective bundles which are trivial in the Euclidean topology which don't come from vector bundles. Consider the exact sequence of algebraic groups,

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{GL}(r) \longrightarrow \mathrm{PGL}(r) \longrightarrow 0.$$

One can sheafify this sequence to get

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathrm{GL}(r) \longrightarrow \mathrm{PGL}(r) \longrightarrow 0.$$

Taking the long exact sequence of cohomology we get

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X, \mathrm{GL}(r)) \longrightarrow H^1(X, \mathrm{PGL}(r)) \longrightarrow H^2(X, \mathcal{O}_X^*).$$

Note that is does make sense to take cohomology of a sheaf of non-abelian groups. Note however that higher cohomology is no longer a group, just a pointed set. The cohomology set

$$H^1(X, \mathrm{GL}(r))$$

classifies vector bundles of rank r. The cohomology set

$$H^1(X, \mathrm{PGL}(r))$$

classifies projective bundles of rank $r-1$. The map between them is the natural map which assigns to a vector bundle the associated projective bundle. The kernel of this map is

$$H^1(X, \mathcal{O}_X^*)$$

which as we have already seen classifies line bundles on X. However the last map

$$H^1(X, \mathrm{PGL}(r)).H^2(X, \mathcal{O}_X^*).$$

is not always zero. The image is the Brauer group; it classifies projective bundles over X which are not Zariski trivial.

There is a fun example of a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. Let

$$Y = \{\, (x, L) \mid x \in L \,\} \subset \mathbb{P}^2 \times \mathbb{P}^2$$

be the incidence correspondence between points and lines on $\mathbb{P}^2$. In coordinates $[x : y : z]$ on the first $\mathbb{P}^2$ and $[a : b : c]$ on the second $\mathbb{P}^2$, Y is given by the bihomogeneous equation

$$ax + by + cz = 0.$$

Consider projection π of Y down to the first $\mathbb{P}^2$. The fibre over a point $[x : y : z]$ is the set of all lines this point. Fix the point $p = [0 : 0 : 1]$. The set of lines through p is given by $c = 0$, so that we get the line $[a : b : 0] \subset \mathbb{P}^2$. Thus the fibres of π are copies of $\mathbb{P}^1$. Now suppose we look at the affine open subset $z \neq 0$ of $\mathbb{P}^2$.

We can use point-slope to see that Y is trivial over $U = \mathbb{A}^2 = (z \neq 0)$. More geometrically, a line through the point $[x : y : z]$ will meet the line L_2, given by $z = 0$, at a unique point. Since a line is specified by two points, it is easy to see that Y is isomorphic to $U \times L_2 \simeq U \times \mathbb{P}^1$.