4. Splitting type

We start with a result due to Grothendieck:

Theorem 4.1 (Grothendieck)**.** *Every vector bundle E on* $\mathbb{P}^1$ *splits as a direct sum of line bundles,*

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(E) \simeq \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i).$$

If we impose the condition $a_i \geq a_{i+1}$ *then the integers* $a_1, a_2, \dots, a_r$ *are unique.*

Proof. We proceed by induction on the rank r of the vector bundle $\mathcal{E}$ on $\mathbb{P}^1$.

We may suppose that $r > 1$ and that the result is true for all smaller values of r. Let

$$d = \inf\{\, k \in \mathbb{Z} \mid H^0(\mathbb{P}^1, \mathcal{E}(k)) \neq 0 \,\}.$$

Note that if k is sufficiently large then $\mathcal{E}(k)$ is globally generated. In particular $d < \infty$. Note also that if $H^0(\mathbb{P}^1, \mathcal{E}(k)) \neq 0$ then there is an injective map of sheaves

$$\mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E}(k)$$

so that

$$\mathcal{O}_{\mathbb{P}^1}(-k) \longrightarrow \mathcal{E}.$$

In particular

$$h^0(\mathbb{P}^1, \mathcal{E}) \geq h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-k)).$$

If $k < 0$ then the LHS is fixed and the RHS goes to ∞ as k gets smaller. Thus $d > -\infty$ and the infimum is a minimum. Note that $\mathcal{E}$ splits if and only if $\mathcal{E}(d)$ splits, so that, replacing $\mathcal{E}$ by $\mathcal{E}(d)$ there is no harm in assuming that $d = 0$. In this case we have

$$H^0(\mathbb{P}^1, \mathcal{E}(-1)) = 0.$$

Pick $\sigma \in H^0(\mathbb{P}^1, \mathcal{E})$. Locally $\mathcal{E}$ is trivial and σ is an r-tuple of holomorphic functions on an open neighbourhood of $0 \in \mathbb{C}$. If σ vanishes at the origin then

$$\sigma(z) = (zf_1(z), zf_2(z), \dots, zf_r(z)).$$

In this case σ defines a section of $\mathcal{I}_p \underset{\mathcal{O}_{\mathbb{P}^1}}{\otimes} \mathcal{E}$. But

$$\mathcal{I}_p \underset{\mathcal{O}_{\mathbb{P}^1}}{\otimes} \mathcal{E} \simeq \mathcal{E}(-1),$$

and so we get a non-zero function

$$\tau \in H^0(\mathbb{P}^1, \mathcal{E}(-1)),$$

which contradicts our choice of d. Thus locally

$$\sigma(z) = (f_1(z), f_2(z), \ldots, f_r(z)),$$

where at least one $f_i(z)$ does not vanish at 0. Thus σ defines an injection

$$\mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E}$$

and the quotient is a locally free sheaf $\mathcal{F}$ of rank $r-1$. Thus we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Twist this exact sequence by $\mathcal{O}_{\mathbb{P}^1}(-1)$ to get the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{E}(-1) \longrightarrow \mathcal{F}(-1) \longrightarrow 0.$$

Consider the long exact sequence of cohomology. The relevant piece is

$$H^0(\mathbb{P}^1, \mathcal{E}(-1)) \longrightarrow H^0(\mathbb{P}^1, \mathcal{F}(-1)) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)).$$

Note that

$$\begin{aligned} h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) &= h^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(1)) \\ &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1-2)) \\ &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \\ &= 0, \end{aligned}$$

by Serre duality. On the other hand

$$h^0(\mathbb{P}^1, \mathcal{E}(-1)) = 0$$

so that

$$h^0(\mathbb{P}^1, \mathcal{F}(-1)) = 0.$$

Now $\mathcal{F}$ is locally free of rank $r-1$. By induction $\mathcal{F}$ is isomorphic to a direct sum of locally free sheaves of rank one,

$$\mathcal{F} \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(a_i),$$

for some integers a_i. Thus

$$\mathcal{F}(-1) \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(a_i - 1).$$

It follows that $a_i - 1 < 0$ for all i, so that $a_i \leq 0$.

Now take the dual of the first exact sequence. We get a short exact sequence of the dual locally free sheaves,

$$0 \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0.$$

I claim that this sequence splits. We have to map the last sheaf back into $\mathcal{E}^*$. It suffices to show that the global section 1 is in the image of the last map; in this case we just send 1 to anything mapping to 1.

Consider the long exact sequence of cohomology. The relevant piece is

$$H^0(\mathbb{P}^1, \mathcal{E}^*) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow H^1(\mathbb{P}^1, \mathcal{F}^*).$$

We have

$$\mathcal{F}^* \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(-a_i).$$

By Serre duality,

$$\begin{aligned} h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a_i)) &= h^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(a_i)) = 0 \\ &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i - 2)) \\ &= 0, \end{aligned}$$

for all i, as $a_i \leq 0$. Thus

$$H^1(\mathbb{P}^1, \mathcal{F}^*) = 0,$$

and so

$$H^0(\mathbb{P}^1, \mathcal{E}^*) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}),$$

is surjective. It follows that the short exact sequence

$$0 \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0,$$

splits. As $\mathcal{F}^*$ splits it follows that $\mathcal{E}^*$ splits and so $\mathcal{E}$ splits. This completes the induction and the proof of the existence of a splitting.

Now suppose that

$$\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i) \simeq \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(b_i),$$

where $a_i \geq a_{i+1}$ and $b_i \geq b_{i+1}$. Suppose that the two sequences are different. Let j be the first index such that $a_j \neq b_j$, so that $a_1 = b_1$, $a_2 = b_2$, ..., $a_{j-1} = b_{j-1}$.

Possibly switching the LHS and the RHS, we may assume that $a_j > b_j$. If we tensor both sides with $-a_j$ and take global sections then the LHS has more sections than the RHS, a contradiction. □

Note that not every sequence of vector bundles on $\mathbb{P}^1$ splits. For example, consider the rank two vector bundle $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. The global section

$$\sigma = (X, Y) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$$

has no zeroes. Therefore we get a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0.$$

Note that the last sheaf is locally free, since σ has no zeroes. It obviously has rank one and it must have degree two, simply by considering the first chern class. If this sequence split then we would have an isomorphism

$$\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2),$$

which contradicts uniqueness of the splitting, (4.1).

It is a theorem in topology that the only topological invariant of a rank r vector bundle on $\mathbb{P}^1$ is its first chern class (the second chern class is zero, as we are on a curve). Note that the first chern class is simply the sum

$$\sum a_i.$$

Thus the topological classification is much coarser than the holomorphic.

Finally, there is a more direct way to argue that

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

splits. One can use global Ext. The obstruction to splitting this sequence lives in

$$\operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^1}) \simeq H^1(\mathbb{P}^1, \mathcal{F}^* \underset{\mathcal{O}_{\mathbb{P}^1}}{\otimes} \mathcal{O}_{\mathbb{P}^1}).$$

To compute the last sheaf, we apply Serre duality as before. For the exact sequence above that does not split, note that splits. One can use global Ext. The obstruction to splitting this sequence lives in

$$\operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(2), \mathcal{O}_{\mathbb{P}^1}) \simeq H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0,$$

as expected.

Note that given a vector bundle on $\mathbb{P}^n$, we can always restrict this vector bundle to a line and consider the splitting type. We carry this out for the tangent bundle on $\mathbb{P}^n$. First observe that the tangent bundle is:

Definition 4.2. *We say that a vector bundle E is **homogeneous** if $\phi^* E \simeq E$ for every element $\phi \in \operatorname{Aut}(\mathbb{P}^n)$.*

Note that a homogeneous vector bundle is **uniform**, meaning that the splitting type is independent of the line, since $\operatorname{Aut}(\mathbb{P}^n)$ is acts transitively on lines.

Note that from the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \longrightarrow T_X \longrightarrow 0,$$

we know that the restriction of the first chern class of the tangent bundle is $n+1$.

We claim that the splitting type is $(2,1,1,\dots,1)$, the most uniform way to distribute the numbers $a_1, a_2, \dots, a_n$ and get the sum $n+1$.

We check the claim. Pick a hyperplane H. The normal bundle of H inside $\mathbb{P}^n$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(1)$ so that there is an exact sequence

$$0 \longrightarrow T_H \longrightarrow T_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0.$$

If we restrict to a line contained in H, the first sheaf becomes

$$\mathcal{O}_{\mathbb{P}^1}(2) \bigoplus_{i=2}^{n-1} \mathcal{O}_{\mathbb{P}^1}(1).$$

and the last sheaf becomes

$$\mathcal{O}_{\mathbb{P}^1}(1).$$

We check this sequence splits. When we compute Ext^1, we have to compute the first cohomology of

$$\mathcal{O}_{\mathbb{P}^1}(1-2) \bigoplus_{i=2}^{n-1} \mathcal{O}_{\mathbb{P}^1}(1-1).$$

But this vanishes, by Serre duality. So the sequence splits and we are done by induction.