5. Rank two vector bundles on $\mathbb{P}^2$

We are going to construct some interesting vector bundles on $\mathbb{P}^2$. First observe the following easy result:

Lemma 5.1. *If E is a rank two vector bundle on $\mathbb{P}^r$ then E splits if and only if E contains a subline bundle.*

Proof. One direction is clear, if E splits there is a surely a subline bundle.

For the other direction, we use the language of sheaves. We may assume that $r > 1$. Let

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^r}(E).$$

If E has a subline bundle then there is an exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \longrightarrow 0,$$

where $\mathcal{L}$ and $\mathcal{M}$ both have rank one. It follows that there are integers a and b such that

$$\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^r}(a) \qquad \text{and} \qquad \mathcal{M} \simeq \mathcal{O}_{\mathbb{P}^r}(b).$$

The obstruction to splitting this exact sequence lies in

$$\operatorname{Ext}^1_{\mathbb{P}^k}(\mathcal{M}, \mathcal{L}) \simeq H^1(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(a-b)) = 0.$$

Thus the exact sequence splits and so E splits. $\square$

Suppose that V is a rank two vector bundle that does not split. If k is large enough, then $V(k)$ is globally generated and we may find a global section σ that has m simple zeroes, $x_1, x_2, \dots, x_m$. Let $\pi\colon X \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at these m points, with exceptional divisors $E_1, E_2, \dots, E_m$. Then $\pi^*\sigma$ is a global section of π^*V which vanishes along each exceptional divisor $E_1, E_2, \dots, E_m$ with multiplicity one,

$$\pi^*\sigma \in H^0(X, \pi^*V \otimes \mathcal{I}_E),$$

where $E = \sum E_i$. Thus we get a subline bundle

$$\mathcal{I}_E^* = \mathcal{O}_X(E) \longrightarrow \sigma^*V.$$

This gives us a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow \sigma^*V \longrightarrow Q \longrightarrow 0,$$

where Q has rank one.

Now $E_1, E_2, \dots, E_m$ are -1-curves, so that $E_i \simeq \mathbb{P}^1$ and $E_i^2 = -1$. Thus when we restrict to E_i we get the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^2 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

Here we identified $Q|_{E_i}$ by using the first chern class.

The idea is simply to reverse all of this. Look for exact sequences

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow V' \longrightarrow \mathcal{O}_X(-E) \longrightarrow 0$$

on X which restrict on each exceptional to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}^2_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

Hopefully V' is then the pullback of some rank two vector bundle V on $\mathbb{P}^2$.

Theorem 5.2. *Let $x_1, x_2, \dots, x_m$ be points of $\mathbb{P}^2$.*

There is a rank two vector bundle V on $\mathbb{P}^2$ such that if $L \subset \mathbb{P}^2$ is a line then

$$V|_L \simeq \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$$

if and only if L contains a points of the m points $x_1, x_2, \dots, x_m$.

Proof. Let $\pi \colon X \longrightarrow \mathbb{P}^2$ blow up the points $x_1, x_2, \dots, x_m$, let E_i be the exceptional divisor over x_i and let $E = \sum E_i$, so that

$$\mathcal{O}_X(E) = \bigotimes_{i=1}^{m} \mathcal{O}_X(E_i).$$

There is a short exact sequence

$$0 \longrightarrow \mathcal{I}_E \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

Now extensions of $\mathcal{O}_X(-E)$ by $\mathcal{O}_X(E)$ are given by

$$\operatorname{Ext}^1_X(\mathcal{O}_X(-E), \mathcal{O}_X(E)) = H^1(X, \mathcal{O}_X(2E))$$

and extensions of $\mathcal{O}_E(-E)$ by $\mathcal{O}_E(E)$ are given by

$$\operatorname{Ext}^1_E(\mathcal{O}_E(-E), \mathcal{O}_E(E)) = H^1(E, \mathcal{O}_E(2E))$$

If we tensor the previous exact sequence by $\mathcal{O}_X(2E)$ then we get the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_X(2E) \longrightarrow \mathcal{O}_E(2E) \longrightarrow 0,$$

Taking the long exact sequence of cohomology we get

$$H^1(X, \mathcal{O}_X(E)) \longrightarrow H^1(X, \mathcal{O}_X(2E)) \longrightarrow H^1(E, \mathcal{O}_E(2E)) \longrightarrow H^2(X, \mathcal{O}_E(E)).$$

Note that

$$\begin{aligned} H^2(X, \mathcal{O}_X(E)) &= H^0(X, \omega_X(-E)) \\ &= H^0(X, \mathcal{O}_X(-K_X - E)). \end{aligned}$$

But $\pi_*(-K_X) = -K_{\mathbb{P}^2} = -3L$, so that the last group is zero (one can also use (5.3) at this point).

It follows that we can lift any extension class on E to an extension on X, so that we can choose a vector bundle on X to be any chosen extension on each E_i. But

$$\operatorname{Ext}^1(\mathcal{O}_E(-E), \mathcal{O}_E(E)) = \bigoplus_{i=1}^{m} \operatorname{Ext}^1(\mathcal{O}_{E_i}(-E_i), \mathcal{O}_{E_i}(E_i)).$$

Let ξ be the extension class, which component by component of E gives

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^2 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

It follows that there is a vector bundle V' on X which is an extension of

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow V' \longrightarrow \mathcal{O}_X(-E) \longrightarrow 0$$

on X which restrict on each exceptional divisor to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^2 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

By (5.4), there is a vector bundle V on $\mathbb{P}^2$ such that $V' = \pi^* V$.

It remains to check the splitting type of V on a line L. Suppose that L is a line that contains the first a points $x_1, x_2, \dots, x_a$ of $x_1, x_2, \dots, x_m$. Let M be the strict transform of L. We have

$$\begin{aligned} L \cdot E &= L \cdot (E_1 + E_2 + \dots + E_m) \\ &= L \cdot (E_1 + E_2 + \dots + E_a) \\ &= a. \end{aligned}$$

If we restrict

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow V' \longrightarrow \mathcal{O}_X(-E) \longrightarrow 0$$

to M we get

$$0 \longrightarrow \mathcal{O}_M(a) \longrightarrow V' \longrightarrow \mathcal{O}_M(-a) \longrightarrow 0.$$

This sequence splits as $a \geq 0$. It follows that

$$\begin{aligned} V|_L &\simeq \sigma^* V|_M \\ &= V'|_M \\ &\simeq \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a). \end{aligned}$$

$\square$

Note that the stratification, into type, of the set of lines in $\mathbb{P}^2$, which is a copy of $\mathbb{P}^2$, decomposes $\mathbb{P}^2$ into a union of locally closed subsets. There is the open subset of lines which avoid $x_1, x_2, \dots, x_m$ and for these the splitting type is $(0, 0)$. There are the lines which meet one point and for these the splitting type is $(1, -1)$. Inside the dual $\mathbb{P}^2$ the lines through a point is a line. Otherwise there are finitely many lines for which the splitting type is of type $(a, -a)$ for $a > 1$.

This part of a more general phenomena, where the splitting type can only jump up, in the lexigraphic order.

Lemma 5.3. *Let $\pi\colon X \longrightarrow Y$ be a birational morphism of smooth varieties.*

Then $R^*\pi_*\mathcal{O}_X = \mathcal{O}_Y$. In particular $H^*(X, \mathcal{O}_X) \simeq H^*(Y, \mathcal{O}_Y)$.

Proof. By weak factorisation, we may assume that π is factored into a sequence of smooth blow ups and smooth blow downs. By induction we may assume that there is one blow up.

There are two ways to proceed.

For the first, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-(j+1)E) \longrightarrow \mathcal{O}_X(-jE) \longrightarrow \mathcal{O}_E(-(j+1)E) \longrightarrow 0.$$

If we take the long exact sequence of cohomology we get

$$R^i\pi_*\mathcal{O}_X(-(j+1)E) \longrightarrow R^i\pi_*\mathcal{O}_X(-jE) \longrightarrow R^i\pi_*\mathcal{O}_E(-jE).$$

Suppose that $j \geq 0$ and $i > 0$. $E \longrightarrow \pi(E)$ is a projective bundle and $\mathcal{O}_E(-jE)$ is relatively nef. Thus the cohomology of $\mathcal{O}_E(-jE)$ restricted to any fibre is zero. Thus $R^i\pi_*\mathcal{O}_E(-jE) = 0$. If j is small enough then the first group vanishes by relative Serre vanishing. Thus the middle group vanishes for all $j \geq 0$ by induction.

Alternatively, we can compute locally. So we may assume that Y is the unit ball in $\mathbb{C}^n$ and π blows up the intersection with a coordinate subspace. The result is then an easy direct computation.

The last statement follows, as the Leray-Serre spectral sequence degenerates at the E_2-level. $\square$

Lemma 5.4. *Let $\pi\colon X \longrightarrow \mathbb{C}^2$ be the blow up of $(0,0) \in \mathbb{C}^2$ with exceptional divisor E, so that*

$$X = \{\, (x, y, [u:v]) \mid [x:y] = [u:v] \,\} \subset \mathbb{C}^2 \times \mathbb{P}^1.$$

If V' is a vector bundle on X which is an extension of

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow V' \longrightarrow \mathcal{O}_X(-E) \longrightarrow 0$$

on X which restrict on the exceptional divisor E to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}^2_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0,$$

then V' is the trivial vector bundle.

In particular there is a vector bundle V on $\mathbb{C}^2$ such that $V' = \pi^*V$.

Proof. Note that the trivial bundle $\mathcal{O}^2_X$ is also an extension

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}^2_X \longrightarrow \mathcal{O}_X(-E) \longrightarrow 0,$$

which restricts to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^2 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

So it suffices to show that

$$\mathrm{Ext}^1(\mathcal{O}_X(-E), \mathcal{O}_X(E)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_E(-E), \mathcal{O}_E(E)),$$

is injective, that is, we have to show

$$H^1(X, \mathcal{O}_X(2E)) \longrightarrow H^1(E, \mathcal{O}_E(2E)),$$

is injective. Considering the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_X(2E) \longrightarrow \mathcal{O}_E(2E) \longrightarrow 0,$$

it suffices to show that

$$H^1(X, \mathcal{O}_X(E)) = 0.$$

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_E(E) \longrightarrow 0.$$

If we take the long exact sequence of cohomology we get

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X(E)) \longrightarrow H^1(E, \mathcal{O}_E(E)).$$

The last group is equal to

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0,$$

which is zero by Serre duality. The first group is trivial, since $H^1(\mathbb{C}, \mathcal{O}_{\mathbb{C}^2}) = 0$ and the blow up won't change H^1 by (5.3). $\square$