6. A splitting criteria

Theorem 6.1. *A vector bundle E on $\mathbb{P}^n$ splits if and only if $H^i(\mathbb{P}^n, E(k)) = 0$ for all $0 < i < n$ and all integers k.*

Proof. One direction is clear, if E splits then we get vanishing, as

$$H^i(\mathbb{P}^n \mathcal{O}_{\mathbb{P}^n}(k)) = 0$$

for all $0 < i < n$.

Now suppose that we have vanishing. We proceed by induction on n. The case $n = 1$ is Grothendieck's result. Otherwise, pick a hyperplane and consider the short exact sequence of sheaves,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

If we tensor this with $\mathcal{E}(k)$ we get

$$0 \longrightarrow \mathcal{E}(k-1) \longrightarrow \mathcal{E}(k) \longrightarrow \mathcal{E}(k)|_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

If we take the long exact sequence of cohomology we get

$$H^i(\mathbb{P}^n, \mathcal{E}(k)) \longrightarrow H^i(\mathbb{P}^{n-1}, \mathcal{E}|_{\mathbb{P}^{n-1}}(k)) \longrightarrow H^{i+1}(\mathbb{P}^n, \mathcal{E}(k-1)).$$

It follows that

$$H^i(\mathbb{P}^{n-1}, \mathcal{E}|_{\mathbb{P}^{n-1}}(k)) = 0,$$

for all $0 < i < n-1$ and every integer k.

It follows by induction that $\mathcal{E}|_{\mathbb{P}^{n-1}}$ splits as a direct sum, so that

$$\mathcal{E}|_{\mathbb{P}^{n-1}} \simeq \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{r-1}}(a_i),$$

for some integers $a_1, a_2, \dots, a_r$. Let

$$\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^r}(a_i).$$

We want to show that $\mathcal{E}$ is isomorphic to $\mathcal{F}$.

Pick an isomorphism

$$\phi\colon \mathcal{F}|_{\mathbb{P}^{n-1}} \longrightarrow \mathcal{E}|_{\mathbb{P}^{n-1}}.$$

Claim 6.2. *We can extend ϕ to a homomorphism*

$$\Phi\colon \mathcal{F} \longrightarrow \mathcal{E}.$$

Proof of (6.2). If we tensor

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

with

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E})$$

then we get the following short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E})(-1) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E})|_{\mathbb{P}^{n-1}} \longrightarrow 0..$$

Note that the last sheaf cohomology group is

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^{n-1}}}(\mathcal{F}|_{\mathbb{P}^{n-1}}, \mathcal{E}|_{\mathbb{P}^{n-1}}).$$

If we take the long exact sequence of cohomology the obstruction to lifting lies in

$$H^1(\mathbb{P}^n, \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E})(-1)).$$

But

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E})(-1) \simeq \mathcal{F}^* \otimes \mathcal{E}(-1),$$

which splits a direct sum of copies twists of $\mathbb{E}$. Thus H^1 vanishes and we can lift ϕ. □

By functoriality, Φ induces a homomorphism

$$\det \Phi \colon \det \mathcal{F} \longrightarrow \det \mathcal{E}.$$

Both line bundles have the same first chern class (since the first chern class is really just a number and this number is determined by its restriction to a hyperplane). Thus we can interpret

$$\begin{aligned} \det \Phi &\in H^0(\mathbb{P}^n, \det \mathcal{F}^* \otimes \det \mathcal{E}) \\ &\in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}). \end{aligned}$$

Thus $\det \Phi$ is a scalar. It restriction to a hyperplane is non-zero and so $\det \Phi$ is a non-zero constant. As $\det \Phi$ is nowhere zero, it follows that Φ is an isomorphism. □

Corollary 6.3. *A vector bundle E on $\mathbb{P}^n$ splits if and only if it restriction to any plane splits.*

Proof. One direction is clear; if E splits its restriction to a plane splits.

For the other direction, pick a hyperplane and consider the short exact sequence of sheaves,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

If we tensor this with $\mathcal{E}(k)$ we get

$$0 \longrightarrow \mathcal{E}(k-1) \longrightarrow \mathcal{E}(k) \longrightarrow \mathcal{E}(k)|_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

By induction we know that $\mathcal{E}(k)|_{\mathbb{P}^{n-1}}$ splits and so by (6.1) we know that

$$H^i(\mathbb{P}^{n-1}, \mathcal{E}(k)|_{\mathbb{P}^{n-1}}) = 0,$$

for every integer k and for all $0 < i < n-1$. If we take the long exact sequence of cohomology we see that

$$H^i(\mathbb{P}^n, \mathcal{E}(k-1)) \longrightarrow H^i(\mathbb{P}^n, \mathcal{E}(k))$$

is surjective for $i < n-1$ and injective for $i > 1$. By Serre vanishing

$$H^i(\mathbb{P}^n, \mathcal{E}(k)) = H^{n-i}(\mathbb{P}^n, \omega_{\mathbb{P}^n} \otimes \mathcal{E}^*(-k))^*$$

for all $0 < i < n$ and for all $|k|$ sufficiently large . Thus

$$H^i(\mathbb{P}^n, \mathcal{E}(k)) = 0,$$

for all k and all $0 < i < n$, by ascending induction if $i < n-1$ and by descending induction if $i > 1$. But then (6.1) implies that E splits. $\square$