7. Uniform bundles

We want to study vector bundles whose splitting type does not depend on the line. Recall that the Grassmannian $\mathbb{G}(1,n)$ parametrises the lines in $\mathbb{P}^n$. The incidence correspondence is the closed subset

$$I = \{\, (p,L) \in \mathbb{P}^n \times \mathbb{G}(1,n) \,|\, p \in L \,\} \subset \mathbb{P}^n \times \mathbb{G}(1,n).$$

There are two natural projections

$$\begin{array}{ccc} I & \xrightarrow{\ g\ } & \mathbb{G}(1,n). \\ {\scriptstyle f}\big\downarrow & & \\ \mathbb{P}^n & & \end{array}$$

The fibre of g over a point $[L]$ is the whole line L. The fibre of f over a point p is the set of lines through p. Pick an auxiliary hyperplane H, not passing through p. Then a line L through intersects H in a unique point q. Vice-versa, given a point $q \in H$ there is a unique line $L = \langle p, q\rangle$ through p and q, and this line meets H in the point q. Thus the fibre over f is a copy of H.

If we look at the subset $G(p) \subset \mathbb{G}(1,n)$ of all lines containing p then the inverse image $B(p)$ of $G(p)$ inside I is almost by definition the blow up of $\mathbb{P}^n$ at the point p.

g realises I as the universal line over the Grassmannian. There is a natural associated rank two vector bundle S to I,

$$S \subset \mathbb{G}(1,n) \times V,$$

where $\mathbb{P}^n = \mathbb{P}(V)$. S is the universal rank two sub bundle of the trivial vector bundle of rank $n+1$ on $\mathbb{G}(1,n)$. I is the projectivisation of S.

Theorem 7.1. *Let E be a vector bundle of rank r on $\mathbb{P}^n$. Fix a point p.*

If $E|_L$ is the trivial vector bundle of rank r for every line L containing p then E is the trivial vector bundle.

Proof. Let $\phi\colon B(p) \longrightarrow \mathbb{P}^n$ be the restriction of f to $B(p)$ so that ϕ is the blow up of p. Let $\gamma\colon B(p) \longrightarrow G(p)$ be the restriction of g to $B(p)$. The fibres of γ are mapped isomorphically by ϕ to lines through p. It follows that ϕ^*E is trivial on the fibres of γ.

Claim 7.2. *There is a vector bundle F on $\mathbb{G}(1,n)$ such that*

$$\phi^*E = \gamma^*F.$$

Assume (7.2). Let D be the exceptional divisor of ϕ. Then the restriction of ϕ^*E to D is the trivial vector bundle of rank r. Let

$i\colon D \longrightarrow B(p)$ be the inclusion of D inside $B(p)$. Then $\gamma \circ i\colon D \longrightarrow G(p)$ is an isomorphism. $(\gamma \circ i)^* F$ is isomorphic to the restriction of F to D, which is the trivial vector bundle of rank r. Therefore $\phi^* E$ is the trivial vector bundle of rank r.

Proof of (7.2). Consider $\gamma_* \phi^* E$. γ is a $\mathbb{P}^1$-bundle. Therefore γ is smooth and so it is flat. Hence $\phi^* E$ is flat over $G(p)$. Let $M = \gamma^{-1} L$. As

$$\begin{aligned} h^0(M, \phi^* E) &= h^0(L, E|_L) \\ &= h^0(L, \mathcal{O}_L^r), \end{aligned}$$

is constant, it follows by the base change theorem that $F = \gamma_* \phi^* E$ is locally free of rank r. Consider the natural map

$$\gamma^* \gamma_* \phi^* E \longrightarrow \phi^* E$$

The first sheaf restricted to M is

$$(\gamma^* \gamma_* \phi^* E)|_M \simeq \mathcal{O}_M \otimes H^0(M, \phi^* E|_M)$$

and this map becomes the evluation map

$$\mathcal{O}_M \otimes H^0(M, \phi^* E|_M) \longrightarrow (\phi^* E)|_M,$$

which is an isomorphism. □

□

Corollary 7.3. *If E is a globally generated bundle on $\mathbb{P}^n$ then E is trivial if and only if $c_1(E) = 0$.*

Proof. One direction is clear; if E is trivial then $c_1(E) = 0$. As E is globally generated, it follows that there is an exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n}^N \longrightarrow E \longrightarrow 0.$$

If we restrict to a line L we get

$$0 \longrightarrow K|_L \longrightarrow \mathcal{O}_L^N \longrightarrow E|_L \longrightarrow 0.$$

If $(a_1, a_2, \dots, a_r)$ is the splitting type, then we see that $a_i \geq 0$, for all i. On the other hand, as the first chern class is zero, we have $\sum a_i = 0$, so that $a_1 = a_2 = \cdots = a_r = 0$. It follows that E is trivial on every line and so we can apply (7.1). □

Lemma 7.4. *Let E be a vector bundle of rank r.*

Then the splitting type is upper semi continuous on $\mathbb{G}(1, n)$.

Proof. It is not hard to see that the splitting type decomposes $\mathbb{G}(1,n)$ into constructible subsets. We just have to show that the splitting type never goes down under specialisation. Suppose that L is a line with splitting type $(a_1, a_2, \dots, a_r)$. By an obvious induction it suffices to prove that the initial part $(a_1, a_2, \dots, a_k)$ is upper semi continuous. We may assume that we are given a curve $C \subset \mathbb{G}(1,n)$ and we may assume that $(a_1, a_2, \dots, a_{k-1})$ is constant. Let a be the generic value of a_k over the curve C. Since global sections of $E(-a)$ can only jump up, it follows that a_k can only jump up. □

Theorem 7.5. *Let E be a uniform vector bundle of rank r on $\mathbb{P}^n$.*
If $r < n$ then E splits.

Proof. We proceed by induction on the rank r. If $r = 1$ there is nothing to prove.

Let $(a_1, a_2, \dots, a_r)$ be the splitting type. Suppose that $a_1 = a_2 = \dots = a_k$ and $a_{k+1} < a_k$. Replacing E by $E(-a_1)$ we may assume that $a_k = 0$. If $k = r$ we may apply (7.1) to conclude that E splits.

Our goal is to find a uniform rank vector bundle F of smaller rank than E and an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow Q \longrightarrow 0,$$

where Q is a uniform vector bundle.

Suppose we can find such a short exact sequence. By induction F and Q split. It follows that $F \otimes Q^*$ splits, so that the extension splits as

$$\operatorname{Ext}^1_{\mathbb{P}^n}(Q, F) \simeq H^1(\mathbb{P}^n, F \otimes Q^*) = 0.$$

To construct F, proceed as before. Consider f^*E. For the fibres M of f, we have

$$f^*E|_M \simeq \mathcal{O}_M^k \oplus E_1,$$

where E_1 is a direct sum of line bundles with negative twists, so that it has no global sections. Thus $\gamma_* f^*E$ is a vector bundle of rank k. As before, consider the natural map

$$g^* g_* f^*E \longrightarrow f^*E$$

The first sheaf restricted to M is

$$(g^* g_* f^*E)|_M \simeq \mathcal{O}_M \otimes H^0(M, f^*E|_M)$$

and this map becomes the evluation map

$$\mathcal{O}_M \otimes H^0(M, f^*E|_M) \longrightarrow (f^*E)|_M,$$

which gives a rank k sub vector bundle. This gives us a short exact sequence of vector bundles on the blow up,

$$0 \longrightarrow F_1 \longrightarrow f^*E \longrightarrow Q_1 \longrightarrow 0,$$

We check that all three bundles are pulled back from $\mathbb{P}^n$. As before it suffices to prove that F_1 and Q_1 are trivial on the fibres of p. Suppose we restrict to $I_p = f^{-1}(p)$. We get an exact sequence

$$0 \longrightarrow F_1|_{I_p} \longrightarrow \mathcal{O}^r_{I_p} \longrightarrow Q_1|_{I_p} \longrightarrow 0,$$

If we take total chern classes we get

$$c(F_1|_{I_p})c(Q_1|_{I_p}) = c(\mathcal{O}^r_{I_p}) = 1.$$

As $r < n$ this forces

$$c(F_1|_{I_p}) = c(Q_1|_{I_p}) = 1.$$

In particular

$$c_1(F_1|_{I_p}) = c_1(Q_1|_{I_p}) = 0.$$

As $Q_1|_{I_p}$ is globally generated, we have $Q_1|_{I_p})$ is a trivial bundle. Taking duals, we get the same conclusion for F_1. Thus they are both pulled back from $\mathbb{P}^n$. □

There is an interesting way to represent uniform vector bundles on $\mathbb{P}^n$. Suppose that E is a uniform bundle on $\mathbb{P}^n$ and suppose that we think of the splitting type as a partition, so that we have pairs (a_i, r_i), where $a_1 > a_2 > a_3 > \dots > a_k$ and E has r_i direct summands of the form $\mathcal{O}_{\mathbb{P}^n}(a_i)$. Then there is a filtration

$$0 = F^0 \subset F^1 \subset F^2 \subset \dots F^k = f^*E$$

of f^*E by sub bundles F^i such that

$$\frac{F^i}{F^{i-1}} = g^*G_i \otimes \mathcal{O}_{\mathbb{P}^n}(a_i)$$

This filtration is called the **Harder-Narasimhan filtration** of f^*E.

It is constructed as follows. Let

$$F^1 = g^*(g_*f^*E(-a_1)) \otimes f^*\mathcal{O}_{\mathbb{P}^n}(a_i).$$

Note that $g_*f^*E(-a_1)$ is a vector bundle of rank k_1 on $\mathbb{G}(1, n)$. As before, F^1 is a sub bundle of f^*E. Let

$$Q_1 = \frac{f^*E}{F^1}$$

be the quotient. The idea is simply to keep going. Let

$$\pi \colon f^*E \longrightarrow Q_1,$$

be the quotient map. Then

$$F^2 = \pi^{-1}(g^* g_*(Q_1(-a_2)) \otimes f^* \mathcal{O}_{\mathbb{P}^n}(a_2))$$

is a sub bundle of f^*E that contains F^1. Form the quotient

$$Q_2 = \frac{f^*E}{F^2}$$

and keep going.

The other direction is much easier. If we have such a filtration, its restriction to a line L defines a sequence of short exact sequences, which must all split.