7. Uniform bundles

We want to study vector bundles whose splitting type does not depend on the line. Recall that the Grassmannian $G(1,n)$ parametrises the lines in $\mathbb{P}^n$. The incidence correspondence is the closed subset

$$I = \{ (p,L) \in \mathbb{P}^n \times G(1,n) \mid p \in L \} \subset \mathbb{P}^n \times G(1,n).$$

There are two natural projections

$$I \overset{g}{\longrightarrow} G(1,n), \quad \mathbb{P}^n \overset{f}{\longrightarrow} I.$$

The fibre of $g$ over a point $[L]$ is the whole line $L$. The fibre of $f$ over a point $p$ is the set of lines through $p$. Pick an auxiliary hyperplane $H$, not passing through $p$. Then a line $L$ through intersects $H$ in a unique point $q$. Vice-versa, given a point $q \in H$ there is a unique line $L = \langle p,q \rangle$ through $p$ and $q$, and this line meets $H$ in the point $q$. Thus the fibre over $f$ is a copy of $H$.

If we look at the subset $G(p) \subset G(1,n)$ of all lines containing $p$ then the inverse image $B(p)$ of $G(p)$ inside $I$ is almost by definition the blow up of $\mathbb{P}^n$ at the point $p$.

$g$ realises $I$ as the universal line over the Grassmannian. There is a natural associated rank two vector bundle $S$ to $I$,

$$S \subset G(1,n) \times V,$$

where $\mathbb{P}^n = \mathbb{P}(V)$. $S$ is the universal rank two sub bundle of the trivial vector bundle of rank $n+1$ on $G(1,n)$. $I$ is the projectivisation of $S$.

**Theorem 7.1.** Let $E$ be a vector bundle of rank $r$ on $\mathbb{P}^n$. Fix a point $p$. 

If $E|_L$ is the trivial vector bundle of rank $r$ for every line $L$ containing $p$ then $E$ is the trivial vector bundle.

**Proof.** Let $\phi: B(p) \rightarrow \mathbb{P}^n$ be the restriction of $f$ to $B(p)$ so that $\phi$ is the blow up of $p$. Let $\gamma: B(p) \rightarrow G(p)$ be the restriction of $g$ to $B(p)$. The fibres of $\gamma$ are mapped isomorphically by $\phi$ to lines through $p$. It follows that $\phi^*E$ is trivial on the fibres of $\gamma$.

**Claim 7.2.** There is a vector bundle $F$ on $G(1,n)$ such that

$$\phi^*E = \gamma^*F.$$

Assume (7.2). Let $D$ be the exceptional divisor of $\phi$. Then the restriction of $\phi^*E$ to $D$ is the trivial vector bundle of rank $r$. Let
\( i: D \rightarrow B(p) \) be the inclusion of \( D \) inside \( B(p) \). Then \( \gamma \circ i: D \rightarrow G(p) \) is an isomorphism. \( (\gamma \circ i)^*F \) is isomorphic to the restriction of \( F \) to \( D \), which is the trivial vector bundle of rank \( r \). Therefore \( \phi^*E \) is the trivial vector bundle of rank \( r \).

\textbf{Proof of (7.2).} Consider \( \gamma_*\phi^*E \). \( \gamma \) is a \( \mathbb{P}^1 \)-bundle. Therefore \( \gamma \) is smooth and so it is flat. Hence \( \phi^*E \) is flat over \( G(p) \). Let \( M = \gamma^{-1}L \).

As 
\[ h^0(M, \phi^*E) = h^0(L, E|_L) = h^0(L, \mathcal{O}_L), \]

is constant, it follows by the base change theorem that \( F = \gamma_*\phi^*E \) is locally free of rank \( r \). Consider the natural map
\[ \gamma^*\gamma_*\phi^*E \rightarrow \phi^*E \]
The first sheaf restricted to \( M \) is 
\[ (\gamma^*\gamma_*\phi^*E)|_M \cong \mathcal{O}_M \otimes H^0(M, \phi^*E|_M) \]
and this map becomes the evaluation map 
\[ \mathcal{O}_M \otimes H^0(M, \phi^*E|_M) \rightarrow (\phi^*E)|_M, \]
which is an isomorphism. \( \square \)

\textbf{Corollary 7.3.} If \( E \) is a globally generated bundle on \( \mathbb{P}^n \) then \( E \) is trivial if and only if \( c_1(E) = 0 \).

\textbf{Proof.} One direction is clear; if \( E \) is trivial then \( c_1(E) = 0 \). As \( E \) is globally generated, it follows that there is an exact sequence 
\[ 0 \rightarrow K \rightarrow \mathcal{O}_{\mathbb{P}^n}^N \rightarrow E \rightarrow 0. \]

If we restrict to a line \( L \) we get 
\[ 0 \rightarrow K|_L \rightarrow \mathcal{O}_L^N \rightarrow E|_L \rightarrow 0. \]

If \( (a_1, a_2, \ldots, a_r) \) is the splitting type, then we see that \( a_i \geq 0 \), for all \( i \). On the other hand, as the first chern class is zero, we have \( \sum a_i = 0 \), so that \( a_1 = a_2 = \cdots = a_r = 0 \). It follows that \( E \) is trivial on every line and so we can apply \( \square \).

\textbf{Lemma 7.4.} Let \( E \) be a vector bundle of rank \( r \).

Then the splitting type is upper semi continuous on \( \mathbb{G}(1, n) \).
Proof. It is not hard to see that the splitting type decomposes $G(1, n)$ into constructible subsets. We just have to show that the splitting type never goes down under specialisation. Suppose that $L$ is a line with splitting type $(a_1, a_2, \ldots, a_r)$. By an obvious induction it suffices to prove that the initial part $(a_1, a_2, \ldots, a_k)$ is upper semi continuous. We may assume that we are given a curve $C \subset G(1, n)$ and we may assume that $(a_1, a_2, \ldots, a_{k-1})$ is constant. Let $a$ be the generic value of $a_k$ over the curve $C$. Since global sections of $E(-a)$ can only jump up, it follows that $a_k$ can only jump up. □

Theorem 7.5. Let $E$ be a uniform vector bundle of rank $r$ on $\mathbb{P}^n$. If $r < n$ then $E$ splits.

Proof. We proceed by induction on the rank $r$. If $r = 1$ there is nothing to prove.

Let $(a_1, a_2, \ldots, a_r)$ be the splitting type. Suppose that $a_1 = a_2 = \cdots = a_k$ and $a_{k+1} < a_k$. Replacing $E$ by $E(-a_1)$ we may assume that $a_k = 0$. If $k = r$ we may apply (7.1) to conclude that $E$ splits.

Our goal is to find a uniform rank vector bundle $F$ of smaller rank than $E$ and an exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow Q \rightarrow 0,$$

where $Q$ is a uniform vector bundle.

Suppose we can find such a short exact sequence. By induction $F$ and $Q$ split. It follows that $F \otimes Q^*$ splits, so that the extension splits as

$$\text{Ext}^1_{\mathbb{P}^n}(Q, F) \simeq H^1(\mathbb{P}^n, F \otimes Q^*) = 0.$$

To construct $F$, proceed as before. Consider $f^* E$. For the fibres $M$ of $f$, we have

$$f^* E|_M \simeq \mathcal{O}_M^k \oplus E_1,$$

where $E_1$ is a direct sum of line bundles with negative twists, so that it has no global sections. Thus $\gamma_* f^* E$ is a vector bundle of rank $k$. As before, consider the natural map

$$g^* g_* f^* E \rightarrow f^* E$$

The first sheaf restricted to $M$ is

$$(g^* g_* f^* E)|_M \simeq \mathcal{O}_M \otimes H^0(M, f^* E|_M)$$

and this map becomes the evaluation map

$$\mathcal{O}_M \otimes H^0(M, f^* E|_M) \rightarrow (f^* E)|_M,$$
which gives a rank $k$ sub vector bundle. This gives us a short exact sequence of vector bundles on the blow up,

$$0 \longrightarrow F_1 \longrightarrow f^*E \longrightarrow Q_1 \longrightarrow 0,$$

We check that all three bundles are pulled back from $\mathbb{P}^n$. As before it suffices to prove that $F_1$ and $Q_1$ are trivial on the fibres of $p$. Suppose we restrict to $I_p = f^{-1}(p)$. We get an exact sequence

$$0 \longrightarrow F_1|_{I_p} \longrightarrow \mathcal{O}_{I_p} \longrightarrow Q_1|_{I_p} \longrightarrow 0,$$

If we take total chern classes we get

$$c(F_1|_{I_p})c(Q_1|_{I_p}) = c(\mathcal{O}_{I_p}) = 1.$$

As $r < n$ this forces

$$c(F_1|_{I_p}) = c(Q_1|_{I_p}) = 1.$$

In particular

$$c_1(F_1|_{I_p}) = c_1(Q_1|_{I_p}) = 0.$$

As $Q_1|_{I_p}$ is globally generated, we have $Q_1|_{I_p}$ is a trivial bundle. Taking duals, we get the same conclusion for $F_1$. Thus they are both pulled back from $\mathbb{P}^n$. □

There is an interesting way to represent uniform vector bundles on $\mathbb{P}^n$. Suppose that $E$ is a uniform bundle on $\mathbb{P}^n$ and suppose that we think of the splitting type as a partition, so that we have pairs $(a_i, r_i)$, where $a_1 > a_2 > a_3 > \cdots > a_k$ and $E$ has $r_i$ direct summands of the form $\mathcal{O}_{\mathbb{P}^n}(a_i)$. Then there is a filtration

$$0 = F^0 \subset F^1 \subset F^2 \subset \cdots F^k = f^*E$$

of $f^*E$ by sub bundles $F^i$ such that

$$\frac{F^i}{F^{i-1}} = g^*G_i \otimes \mathcal{O}_{\mathbb{P}^n}(a_i)$$

This filtration is called the **Harder-Narasimhan filtration** of $f^*E$.

It is constructed as follows. Let

$$F^1 = g^*(g_*f^*E(-a_1)) \otimes f^*\mathcal{O}_{\mathbb{P}^n}(a_i).$$

Note that $g_*f^*E(-a_1)$ is a vector bundle of rank $k_1$ on $\mathbb{G}(1, n)$. As before, $F^1$ is a sub bundle of $f^*E$. Let

$$Q_1 = \frac{f^*E}{F^1}$$

be the quotient. The idea is simply to keep going. Let

$$\pi : f^*E \longrightarrow Q_1,$$
be the quotient map. Then

\[ F^2 = \pi^{-1}(g^*g_* (Q_1(-a_2)) \otimes f^*\mathcal{O}_{\mathbb{P}^n}(a_2)) \]

is a sub bundle of \( f^*E \) that contains \( F^1 \). Form the quotient

\[ Q_2 = \frac{f^*E}{F^2} \]

and keep going.

The other direction is much easier. If we have such a filtration, its restriction to a line \( L \) defines a sequence of short exact sequences, which must all split.