8. Uniform heterogeneous examples

We are going to give an example of a bundle which is uniform, meaning that the splitting type is constant, but not homogeneous, so that the bundle is not fixed under the action of the automorphism group of $\mathbb{P}^n$.

Definition 8.1. *We say that a vector bundle E is k-**homogeneous** if $\phi_1^* E \simeq \phi_2^* E$ for all linear maps $\phi_1 \colon \mathbb{P}^k \longrightarrow \mathbb{P}^n$ and $\phi_2 \colon \mathbb{P}^k \longrightarrow \mathbb{P}^n$.*

Note that E is homogeneous if and only if it is n-homogeneous and it is uniform if and only if it is 1-homogeneous. Since every linear map can be extended from $\mathbb{P}^k$ to $\mathbb{P}^{k+1}$, for $k < n$ it follows that $(k+1)$-homogeneous implies k-homogeneous.

Definition 8.2. *The maximum k such that E is k-homogeneous, denoted $h(E)$, is called the **extent** of E.*

If the rank r of E is less than n, $r < n$, then either E splits, in which case E is homogeneous or E is not uniform. Thus

$$h(E) = 0 \qquad \text{or} \qquad h(E) = n,$$

when $r < n$.

Theorem 8.3. *Let $n \neq 2$. For every $0 \leq e < n - 1$ there is a holomorphic vector bundle E on $\mathbb{P}^n$ with extent e.*

Proof. We start with the Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow 0.$$

The first bundle is the universal sub line bundle. So the bundle

$$T_{\mathbb{P}^n}(-1)$$

is the quotient of the trivial bundle $\mathbb{P}^n \times V$ by the universal sub line bundle S, where $\mathbb{P}^n = \mathbb{P}(V)$.

Pick a basis $w_0, w_1, \dots, w_n$ of V. These determine sections

$$s_i \in H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-1))$$

which at the point p takes the value w_i/S_p.

As the $w_0, w_1, \dots, w_m$ are linearly independent, it follows that $s_0, s_1, \dots, s_m$ have no common zeroes. Thus we get an inclusion

$$\mathcal{O}_{\mathbb{P}^n} \longrightarrow T_{\mathbb{P}^n}(-1)^{\oplus m+1}.$$

Let E be the quotient vector bundle, so that there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow T_{\mathbb{P}^n}(-1)^{\oplus m+1} \longrightarrow E \longrightarrow 0.$$

Note that E has rank

$$(m+1)n-1.$$

Consider what happens if we restrict E to a linear subspace $\mathbb{P}(W) \subset \mathbb{P}(V)$.

Claim 8.4. *Let W_0 be the span of the vectors $w_0, w_1, \dots, w_m$, let $\Lambda_0 = \mathbb{P}(W_0)$ and let $\Lambda \subset \mathbb{P}^n$ be a k-dimensional linear subspace.*

(1) If Λ_0 is not contained in Λ, then

$$E|_\Lambda \simeq T_\Lambda(-1)^{\oplus m+1} \oplus \mathcal{O}_\Lambda^{\oplus (n-k)(m+1)-1}.$$

(2) If Λ_0 is contained in Λ, then

$$E|_\Lambda \simeq E' \oplus \mathcal{O}_\Lambda^{\oplus (n-k)(m+1)},$$

where E' is a bundle on Λ such that $h^0(\Lambda, E^*) = 0$.

If we assume the claim then note that

$$h^0(\Lambda, E^*|_\Lambda) = \begin{cases} (n-k)(m+1)-1 & \text{otherwise} \\ (n-k)(m+1) & \text{if } \Lambda_0 \subset \Lambda. \end{cases}$$

In particular if $k = m-1$ then we are always in the first case, so that E is $(m-1)$-homogeneous. If $k = m$ then there are two possibilities for $h^0(\Lambda, E|_\Lambda)$ so that the E is not m-homogeneous. Thus the extent of E is $m-1$.

Proof of (8.4). First suppose that Λ_0 is not contained in Λ. Then we may assume that w_0 is not contained in W. It follows that the restriction of

$$s_0 \in H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-1))$$

to Λ is nowhere zero. s_0 defines a short exact sequence

$$0 \longrightarrow \mathcal{O}_\Lambda \longrightarrow T_{\mathbb{P}^n}(-1))|_\Lambda \longrightarrow Q \longrightarrow 0,$$

where Q is a vector bundle of rank $n-1$ on Λ. There is also an exact sequence

$$0 \longrightarrow T_\Lambda(-1) \longrightarrow T_{\mathbb{P}^n}(-1)|_\Lambda \longrightarrow {\mathcal{O}_\Lambda}^{\oplus(n-k)} \longrightarrow 0.$$

This gives a commutative diagram with exact rows and colummns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_\Lambda(-1) & = & T_\Lambda(-1) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_\Lambda & \to & T_{\mathbb{P}^n}(-1)|_\Lambda & \to & Q & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_\Lambda & \to & \mathcal{O}_\Lambda^{\oplus n-k} & \to & Q' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

The bottom row yields the isomorphism

$$Q' \simeq \mathcal{O}_\Lambda^{\oplus n-k-1}$$

and so the right column gives the isomorphism

$$Q \simeq T_\Lambda(-1) \oplus \mathcal{O}_\Lambda^{\oplus n-k-1}$$

since

$$H^1(\Lambda, T_\Lambda(-1)) = 0.$$

We now consider another similar diagram.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_{\mathbb{P}^n}(-1)^{\oplus m}|_\Lambda & = & T_{\mathbb{P}^n}(-1)^{\oplus m}|_\Lambda & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_\Lambda & \to & T_{\mathbb{P}^n}(-1)^{\oplus m+1}|_\Lambda & \to & E|_\Lambda & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_\Lambda & \to & T_{\mathbb{P}^n}(-1)|_\Lambda & \to & Q & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Now the last column splits as

$$H^1(\Lambda, Q^* \otimes T_\Lambda(-1)^{\oplus m}|_\Lambda) = 0.$$

It follows then that

$$\begin{aligned} E|_\Lambda &= T_{\mathbb{P}^n}(-1)^{\oplus m}|_\Lambda \oplus T_\Lambda(-1) \oplus \mathcal{O}_\Lambda^{\oplus n-k-1} \\ &= T_\Lambda(-1)^{\oplus m+1}|_\Lambda \oplus \mathcal{O}_\Lambda^{\oplus (n-k)(m+1)-1}. \end{aligned}$$

Now suppose that $\Lambda_0 \subset \Lambda$.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_\Lambda & \to & T_\Lambda(-1)^{\oplus m+1} & \to & E' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_\Lambda & \to & T_{\mathbb{P}^n}(-1)^{\oplus m+1}|_\Lambda & \to & E|_\Lambda & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{O}_\Lambda^{\oplus (n-k)(m+1)} & = & \mathcal{O}_\Lambda^{\oplus (n-k)(m+1)} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

From the top row we get

$$\begin{aligned} h^0(\Lambda, E'^*) &= 0 \\ h^1(\Lambda, E') &= 0. \end{aligned}$$

From the last column we then deduce

$$E|_\Lambda \simeq E' \oplus \mathcal{O}_\Lambda^{\oplus (n-k)(m+1)}. \qquad \square$$

$\square$

If we take $m = 2$ then the rank of E is $3n - 1$ and E is not homogeneous but it is uniform.