8. Uniform heterogeneous examples

We are going to give an example of a bundle which is uniform, meaning that the splitting type is constant, but not homogeneous, so that the bundle is not fixed under the action of the automorphism group of \( \mathbb{P}^n \).

**Definition 8.1.** We say that a vector bundle \( E \) is \( k \)-homogeneous if \( \phi_1^* E \simeq \phi_2^* E \) for all linear maps \( \phi_1 : \mathbb{P}^k \to \mathbb{P}^n \) and \( \phi_2 : \mathbb{P}^k \to \mathbb{P}^n \).

Note that \( E \) is homogeneous if and only if it is \( n \)-homogeneous and it is uniform if and only if it is 1-homogeneous. Since every linear map can be extended from \( \mathbb{P}^k \) to \( \mathbb{P}^{k+1} \), for \( k < n \) it follows that \((k+1)\)-homogeneous implies \( k \)-homogeneous.

**Definition 8.2.** The maximum \( k \) such that \( E \) is \( k \)-homogeneous, denoted \( h(E) \), is called the extent of \( E \).

If the rank \( r \) of \( E \) is less than \( n \), \( r < n \), then either \( E \) splits, in which case \( E \) is homogeneous or \( E \) is not uniform. Thus
\[
h(E) = 0 \quad \text{or} \quad h(E) = n,
\]
when \( r < n \).

**Theorem 8.3.** Let \( n \neq 2 \). For every \( 0 \leq e < n - 1 \) there is a holomorphic vector bundle \( E \) on \( \mathbb{P}^n \) with extent \( e \).

**Proof.** We start with the Euler sequence,
\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow 0.
\]
The first bundle is the universal sub line bundle. So the bundle
\[
T_{\mathbb{P}^n}(-1)
\]
is the quotient of the trivial bundle \( \mathbb{P}^n \times V \) by the universal sub line bundle \( S \), where \( \mathbb{P}^n = \mathbb{P}(V) \).

Pick a basis \( w_0, w_1, \ldots, w_n \) of \( V \). These determine sections
\[
s_i \in H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-1))
\]
which at the point \( p \) takes the value \( w_i / S_p \).

As the \( w_0, w_1, \ldots, w_m \) are linearly independent, it follows that \( s_0, s_1, \ldots, s_m \) have no common zeroes. Thus we get an inclusion
\[
\mathcal{O}_{\mathbb{P}^n} \longrightarrow T_{\mathbb{P}^n}(-1)^{\oplus m+1}.
\]
Let \( E \) be the quotient vector bundle, so that there is a short exact sequence
\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow T_{\mathbb{P}^n}(-1)^{\oplus m+1} \longrightarrow E \longrightarrow 0.
\]
Note that $E$ has rank 

$$(m + 1)n - 1.$$ 

Consider what happens if we restrict $E$ to a linear subspace $\mathbb{P}(W) \subset \mathbb{P}(V)$.

**Claim 8.4.** Let $W_0$ be the span of the vectors $w_0, w_1, \ldots, w_m$, let $\Lambda_0 = \mathbb{P}(W_0)$ and let $\Lambda \subset \mathbb{P}^n$ be a $k$-dimensional linear subspace.

1. If $\Lambda_0$ is not contained in $\Lambda$, then 

$$E|_\Lambda \simeq T_\Lambda(-1)^{\oplus m+1} \oplus \mathcal{O}_\Lambda^{\oplus (n-k)(m+1)-1}.$$ 

2. If $\Lambda_0$ is contained in $\Lambda$, then 

$$E|_\Lambda \simeq E' \oplus \mathcal{O}_\Lambda^{\oplus (n-k)(m+1)},$$ 

where $E'$ is a bundle on $\Lambda$ such that $h^0(\Lambda, E^*) = 0$.

If we assume the claim then note that 

$$h^0(\Lambda, E^*|_\Lambda) = \begin{cases} (n - k)(m + 1) - 1 & \text{otherwise} \\ (n - k)(m + 1) & \text{if } \Lambda_0 \subset \Lambda. \end{cases}$$

In particular if $k = m - 1$ then we are always in the first case, so that $E$ is $(m - 1)$-homogeneous. If $k = m$ then there are two possibilities for $h^0(\Lambda, E|_\Lambda)$ so that the $E$ is not $m$-homogeneous. Thus the extent of $E$ is $m - 1$.

**Proof of Claim 8.4.** First suppose that $\Lambda_0$ is not contained in $\Lambda$. Then we may assume that $w_0$ is not contained in $W$. It follows that the restriction of 

$$s_0 \in H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-1))$$

to $\Lambda$ is nowhere zero. $s_0$ defines a short exact sequence

$$0 \rightarrow \mathcal{O}_\Lambda \rightarrow T_{\mathbb{P}^n}(-1)|_\Lambda \rightarrow Q \rightarrow 0,$$

where $Q$ is a vector bundle of rank $n - 1$ on $\Lambda$. There is also an exact sequence

$$0 \rightarrow T_\Lambda(-1) \rightarrow T_{\mathbb{P}^n}(-1)|_\Lambda \rightarrow \mathcal{O}_\Lambda^{\oplus (n-k)} \rightarrow 0.$$
This gives a commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
T_{\Lambda}(-1) = T_{\Lambda}(-1) \\
\downarrow & \downarrow \\
0 \to O \to \Lambda \to T_{\mathbb{P}^n}(1)|_{\Lambda} \to Q \to 0 \\
\parallel & \downarrow & \downarrow \\
0 \to O \to O_{\Lambda} \oplus m \to \Lambda \to Q' \to 0 \\
\downarrow & \downarrow \\
0 & 0 \\
\end{array}
\]

The bottom row yields the isomorphism

\[Q' \simeq O_{\Lambda}^\oplus n-k-1\]

and so the right column gives the isomorphism

\[Q \simeq T_{\Lambda}(-1) \oplus O_{\Lambda}^\oplus n-k-1\]

since

\[H^1(\Lambda, T_{\Lambda}(-1)) = 0.\]

We now consider another similar diagram.

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
T_{\mathbb{P}^n}(1)^{\oplus m}|_{\Lambda} = T_{\mathbb{P}^n}(1)^{\oplus m}|_{\Lambda} \\
\downarrow & \downarrow \\
0 \to O_{\Lambda} \to T_{\mathbb{P}^n}(1)^{\oplus m+1}|_{\Lambda} \to E|_{\Lambda} \to 0 \\
\parallel & \downarrow & \downarrow \\
0 \to O_{\Lambda} \to T_{\mathbb{P}^n}(1)|_{\Lambda} \to Q \to 0 \\
\downarrow & \downarrow \\
0 & 0 \\
\end{array}
\]

Now the last column splits as

\[H^1(\Lambda, Q^* \otimes T_{\Lambda}(1)^{\oplus m}|_{\Lambda}) = 0.\]
It follows then that
\[
E|_\Lambda = T_{\mathbb{P}^n}(-1)^{\oplus m}|_\Lambda \oplus T_\Lambda(-1) \oplus O_\Lambda^{\oplus n-k-1} \\
= T_\Lambda(-1)^{\oplus m+1}|_\Lambda \oplus O_\Lambda^{\oplus (n-k)(m+1)-1}.
\]

Now suppose that \( \Lambda_0 \subset \Lambda \).

\[
\begin{array}{cccccc}
0 & 0 & \downarrow & \downarrow \\
\| & \downarrow & \downarrow \\
0 \rightarrow O_\Lambda \rightarrow T_\Lambda(-1)^{\oplus m+1} \rightarrow E' \rightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 \rightarrow O_\Lambda \rightarrow T_{\mathbb{P}^n}(-1)^{\oplus m+1}|_\Lambda \rightarrow E|_\Lambda \rightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
O_\Lambda^{\oplus (n-k)(m+1)} = O_\Lambda^{\oplus (n-k)(m+1)} \\
\downarrow & \downarrow & \downarrow \\
0 & 0
\end{array}
\]

From the top row we get
\[
h^0(\Lambda, E'^*) = 0 \\
h^1(\Lambda, E') = 0.
\]
From the last column we then deduce
\[
E|_\Lambda \simeq E' \oplus O_\Lambda^{\oplus (n-k)(m+1)}.
\]

If we take \( m = 2 \) then the rank of \( E \) is \( 3n - 1 \) and \( E \) is not homogeneous but it is uniform.