9. Simple bundles

Definition 9.1. *A vector bundle is called **simple** if*
$$h^0(\mathbb{P}^n, E^* \otimes E) = 1.$$

As
$$E^* \otimes E = \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(E, E),$$
a bundle is simple if and only if its only endomorphisms are homotheties.

Definition 9.2. *A vector bundle E on $\mathbb{P}^n$ is **decomposable** if it is isomorphic to a direct sum $F \oplus G$, where F and G have smaller rank than E.*

If E is decomposable then it has non-trivial endomorphisms, given by different homotheties on both factors. Thus simple bundles are always indecomposable.

Lemma 9.3. *The tangent bundle on $\mathbb{P}^n$ is simple.*

Proof. We start with the Euler sequence,
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow 0.$$
If we tensor this with $\Omega^1_{\mathbb{P}^n}(1)$ then we get the exact sequence
$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \Omega^1_{\mathbb{P}^n}(1)^{\oplus n+1} \longrightarrow \Omega^1_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n} \longrightarrow 0.$$
Taking the long exact sequence of cohomology we get
$$H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(1)^{\oplus n+1}) \longrightarrow H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) \longrightarrow H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n})$$
Now
$$h^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(1)^{\oplus n+1}) = 0 \qquad \text{and} \qquad h^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) = 1.$$
But then
$$1 \leq h^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) \leq h^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) = 1.$$
□

It follows that the bundles $T_{\mathbb{P}^n}(k)$ and $\Omega^1_{\mathbb{P}^n}(k)$ are all simple, for any integer k.

Consider the rank two bundles we constructed on $\mathbb{P}^2$. We started with points in $p_1, p_2, \dots, p_k$ in $\mathbb{P}^2$, looked at the surface $\pi \colon X \longrightarrow \mathbb{P}^2$ you get by blowing up these points, and modified the extension
$$0 \longrightarrow \mathcal{O}_X(C) \longrightarrow E' \longrightarrow \mathcal{O}_X(-C) \longrightarrow 0,$$
where C is the sum of the exceptionals, so that E' is the pullback of a bundle E on $\mathbb{P}^2$, $E' = \pi^* E$.

Recall that if L is a line that contains a points then the splitting type is $(a, -a)$. In particular E is indecomposable. We want to show that E is not simple. We will need:

Lemma 9.4. *If E is a vector bundle of rank $r > 1$ and both $h^0(\mathbb{P}^n, E) > 0$ and $h^0(\mathbb{P}^n, E^*) > 0$ then E is not simple.*

Proof. Pick non-zero sections

$$\sigma \in H^0(\mathbb{P}^n, E) \qquad \text{and} \qquad \tau \in H^0(\mathbb{P}^n, E^*).$$

Then

$$\sigma \otimes \tau \in H^0(\mathbb{P}^n, E^* \otimes E)$$

is not a homothety, as it has rank one when restricted to any fibre. □

Note that

$$\begin{aligned} E^* &\simeq E \otimes \det E^* \\ &\simeq E, \end{aligned}$$

since $c_1(E) = 0$. On the other hand,

$$h^0(\mathbb{P}^2, E) > 0,$$

by construction, since E' contains the sub line bundle $\mathcal{O}_X(C)$.

We want to construct a simple bundle N of rank $n-1$ on $\mathbb{P}^n$, for any odd integer n, called the **null correlation bundle**. We will construct N as the kernel of a surjective map

$$T_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1).$$

We get a short exact sequence

$$0 \longrightarrow N \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0.$$

We first check that N is simple. We tensor the above sequence by N^* to get

$$h^0(\mathbb{P}^n, N^* \otimes N) \leq h^0(\mathbb{P}^n, N^* \otimes T_{\mathbb{P}^n}(-1)).$$

Now tensor the dual of the short exact sequence above with $T_{\mathbb{P}^n}(-1)$ to get

$$0 \longrightarrow T_{\mathbb{P}^n}(-2) \longrightarrow T_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n} \longrightarrow N^* \otimes T_{\mathbb{P}^n}(-1) \longrightarrow 0.$$

If we take the associated long exact sequence of cohomology we get

$$H^0(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n}) \longrightarrow H^0(\mathbb{P}^n, N^* \otimes T_{\mathbb{P}^n}(-1)) \longrightarrow H^1(\mathbb{P}^n, T_{\mathbb{P}^n}(-2)).$$

Now

$$H^1(\mathbb{P}^n, T_{\mathbb{P}^n}(-2)) = 0,$$

and so

$$h^0(\mathbb{P}^n, N^* \otimes T_{\mathbb{P}^n}(-1)) \leq h^0(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n}) = 1.$$

Thus

$$1 \leq h^0(\mathbb{P}^n, N^* \otimes N) \leq h^0(\mathbb{P}^n, N^* \otimes T_{\mathbb{P}^n}(-1)) \leq 1.$$

Now consider the goal of finding a surjective map

$$T_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1).$$

Dualising, it suffices to find a nowhwere zero section of $\Omega^1_{\mathbb{P}^n}(2)$.

Consider the projective bundle

$$\mathbb{P}(\Omega^1_{\mathbb{P}^n})$$

over $\mathbb{P}^n$. Point in this bundle correspond to hyperplanes in the tangent bundle, so that

$$\mathbb{P}(\Omega^1_{\mathbb{P}^n}) \simeq \{\, (x, H) \,|\, x \in H \,\} \subset \mathbb{P}^n \times \mathbb{P}^n.$$

Let

$$p\colon \mathbb{P}(\Omega^1_{\mathbb{P}^n}) \longrightarrow \mathbb{P}^n$$

be the natural projection. As n is odd, $n+1 = 2m$ for some integer m. Let A be the block diagonal $2m \times 2m$ matrix with the 2×2 matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then A is invertible and for all $x \in \mathbb{C}^{n+1}$ we have

$$\langle Ax, x \rangle = 0,$$

where

$$\langle x, y \rangle = \sum x_i y_i,$$

is the standard inner product. Pick homogeneous coordinates $[x_0 : x_1 : \dots : x_n]$ on $\mathbb{P}^n$ and $[\xi_0 : \xi_1 : \dots : \xi_n]$ on the dual $\mathbb{P}^n$. In these coordinates, A defines an isomorphism

$$\Phi\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n$$

such that $x \in H = \Phi(x)$. The graph of Φ defines a section

$$g\colon \mathbb{P}^n \longrightarrow \mathbb{P}(\Omega^1_{\mathbb{P}^n}),$$

which sends x to $(x, \Phi(x))$. This gives a sub line bundle of $\Omega^1_{\mathbb{P}^n}$,

$$\mathcal{O}_{\mathbb{P}^n}(a) \longrightarrow \Omega^1_{\mathbb{P}^n}.$$

Lemma 9.5. $a = -2$.

Proof. Note that

$$\Omega^1_{\mathbb{P}^n}(-a)$$

has a non-vanishing section and so

$$c_n(\Omega^1_{\mathbb{P}^n}(-a)) = 0.$$

We compute

$$
\begin{aligned}
0 &= c_n(\Omega^1_{\mathbb{P}^n}(-a)) \\
&= -c_n(T_{\mathbb{P}^n}(a)) \\
&= -\sum_{i=0}^{n} c_i(T_{\mathbb{P}^n}) a^{n-i} \\
&= -\sum_{i=0}^{n} \binom{n+1}{i} a^{n-i}.
\end{aligned}
$$

If we multiply by a we get

$$
\begin{aligned}
0 &= \sum_{i=0}^{n} \binom{n+1}{i} a^{n+1-i} \\
&= (1+a)^{n+1} - 1.
\end{aligned}
$$

It follows that $1 + a = \pm 1$. As $c_n(T_{\mathbb{P}^n}) = n + 1 \neq 0$, we cannot be in the case $a = 0$. Thus $a = -2$. □

Putting all of this together, we can construct the null correlation bundle N.

Note that

$$
\begin{aligned}
c(N) &= \frac{c(T_{\mathbb{P}^n}(-1))}{1+h} \\
&= \frac{1}{(1+h)(1-h)} \\
&= 1 + h^2 + h^4 + h^6 + \cdots + h^{n-1}.
\end{aligned}
$$

Thus we have proved:

Theorem 9.6. *For every odd integer n there is a simple bundle N on $\mathbb{P}^n$ with total chern class:*

$$c(N) = 1 + h^2 + h^4 + h^6 + \cdots + h^{n-1}.$$