15. Farey Sequences

**Definition 15.1.** Fix a natural number $n$.

The **Farey sequence** of order $n$, denoted $F_n$, is the set of rational numbers $p/q$ with denominator $1 \leq q \leq n$, arranged in increasing order.

$F_1$ is the sequence of integers, and so on.

**Lemma 15.2.** If $n$ is a natural number then

$$|F_n \cap [0, 1]| = 1 + \varphi(1) + \varphi(2) + \cdots + \varphi(n).$$

**Proof.** Indeed an element of

$$F_n \cap (0, 1)$$

has the unique form $a/b$, where $2 \leq b \leq n$ and $a$ is coprime to $b$. \qed

**Definition 15.3.** $p/q$ and $r/s \in F_n$ are **adjacent** if they are successive elements of the sequence $F_n$.

**Definition-Proposition 15.4.**

(1) If $p/q$ and $r/s$ are adjacent in $F_n$ for some $n$ then $|ps - qr| = 1$.

(2) If $|ps - qr| = 1$ then $p/q$ and $r/s$ are adjacent in $F_n$ for

$$\text{max}(q, s) \leq n < q + s.$$ and they are separated by the single element

$$\left(\frac{p + r}{q + s}\right) \in F_{q+s},$$

called the **mediant** of $p/q$ and $r/s$.

**Proof.** We first prove (2). Suppose that $p/q$ and $r/s$ are two elements of $F_n$ such that $qr - ps = \pm 1$. Possibly switching $p/q$ and $r/s$ we may assume that $r/s > p/q$ and $qr - ps = 1$.

Consider the function

$$f : [0, \infty] \rightarrow [p/q, r/s] \quad \text{given by} \quad f(t) = \frac{p + tr}{q + ts}.$$ As $t$ increases from 0 to $\infty$, $f$ increases from $p/q$ to $r/s$. Thus $f$ is a bijection. Moreover it is clear that $f(t)$ is rational if and only if $t$ is rational. Thus we may assume that $t = u/v$, where $u, v > 0$ and $(u, v) = 1$. We have

$$f\left(\frac{u}{v}\right) = \frac{vp + ur}{vq + us}.$$
As
\[ q(vp + ur) - p(vq + us) = u(qr - ps) = u \\
    s(vp + ur) - r(vq + us) = v(ps - qr) = -v, \]
it follows that \( vp + ur \) is coprime to \( vq + us \).

It follows that as \( u \) and \( v \) run over all coprime integers, \( f(u/v) \) runs over all rational numbers between \( p/q \) and \( r/s \). Amongst all such choices, \( u = v = 1 \) gives the smallest denominator. \( f(1) \) is the mediant of \( p/q \) and \( r/s \) and for future reference note that
\[ |(p + r)q - (q + s)p| = 1 \quad \text{and} \quad |r(q + s) - s(p + r)| = 1. \]
Since \( q + s > \max(q, s) \), (2) holds.

We now turn to (1). We prove this by induction on \( n \). If \( n = 1 \) then \( p/q = a/1 \) and \( r/s = (a + 1)/1 \) so that
\[ |ps - qr| = |a \cdot 1 - (a + 1) \cdot 1| = 1. \]
Thus (1) holds when \( n = 1 \).

Now suppose that (1) holds for \( n \). The only elements of \( \mathcal{F}_{n+1} \) not in \( \mathcal{F}_n \) are mediants of elements of \( \mathcal{F}_n \) and we have already checked (1) in this case. Thus (1) holds by induction. \( \square \)

**Theorem 15.5 (Hurwitz).** Suppose that the real number \( x \) is between two adjacent elements \( r/s \) and \( u/v \) of \( \mathcal{F}_n \).

Then at least one of the three numbers
\[ \frac{r}{s}, \frac{u}{v} \quad \text{and} \quad \frac{l}{m} = \frac{(r + u)}{(s + v)} \]
is a solution of the equation
\[ \left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}. \]

In particular if \( x \) is irrational then we may find infinitely many such \( p \) and \( q \).

**Proof.** Possibly relabelling, we may assume that
\[ \frac{r}{s} < \frac{l}{m} < \frac{u}{v}. \]
If \( p/q \) is one of these three numbers and \( c \) is a positive real number then let \( I_c(p/q) \) be the interval
\[ \left[ \frac{p}{q} - \frac{1}{cq^2}, \frac{p}{q} + \frac{1}{cq^2} \right]. \]
We want to find the largest value of \( c \) so that the three intervals \( I_c(r/s) \), \( I_c(l/m) \) and \( I_c(u/v) \) completely cover the interval

\[
I = \left[ \frac{r}{s}, \frac{u}{v} \right].
\]

Note that \( I_c(r/s) \) intersects \( I_c(u/v) \) if

\[
\frac{r}{s} + \frac{1}{cs^2} \geq \frac{u}{v} - \frac{1}{cv^2}.
\]

Rearranging, this gives

\[
\frac{1}{c} \left( \frac{1}{s^2} + \frac{1}{v^2} \right) \geq \frac{u}{v} - \frac{r}{s} = \frac{1}{vs},
\]

so that

\[
c \leq \frac{v}{s} + \frac{s}{v}.
\]

If we let

\[
f(t) = t + \frac{1}{t}
\]

then

\[
c \leq f\left(\frac{v}{s}\right).
\]

By a similar analysis, \( I_c(r/s) \) and \( I_c(l/m) \) intersect if

\[
c \leq f\left(\frac{m}{s}\right) = f\left(1 + \frac{v}{s}\right).
\]

Consider the problem of trying to cover the left-hand portion

\[
\left[ \frac{r}{s}, \frac{l}{m} \right]
\]

of the interval \( I \) by the union \( I_c \) of the three intervals. \( I \) is covered by \( I_c \) if either of these intervals intersect, that is, we are done if

\[
c \leq \max \left( f\left(\frac{v}{s}\right), f\left(1 + \frac{v}{s}\right) \right).
\]

So we are definitely done if

\[
c \leq \min_{t > 0} \max \left( f(t), f(1 + t) \right)
\]

since we are taking a minimum over values that include

\[
t = \frac{v}{s}.
\]

The minimum occurs for that value \( t_0 \) of \( t \) for which \( f(t) = f(1 + t) \). This gives the equation

\[
t + \frac{1}{t} = t + 1 + \frac{1}{1 + t}.
\]
Cancelling the \( t \) and cross-multiplying, it follows that
\[
1 + t = t(1 + t) + t.
\]
Thus
\[
t^2 + t - 1 = 0.
\]
The positive root of this equation is
\[
t_0 = \frac{\sqrt{5} - 1}{2}.
\]
It follows that
\[
c_0 = \sqrt{5}.
\]
By symmetry the right-hand portion
\[
\left[ \frac{l}{m}, \frac{u}{v} \right]
\]
is also covered if \( c \leq \sqrt{5} \).

Thus \( I \) belongs to the union \( I_c \) of the intervals if \( c \leq \sqrt{5} \). This shows we get an inequality
\[
\left| x - \frac{p}{q} \right| \leq \frac{1}{\sqrt{5}q^2}.
\]
As \( \sqrt{5} \) is irrational, we must in fact have strict inequality.

If \( x \) is irrational the interval determined by adjacent points of \( F_n \) to which \( x \) belongs must shrink down to \( x \), on both sides of \( x \). Thus we get infinitely many \( p/q \) this way.
\[\square\]