FINAL EXAM
MATH 104B, UCSD, WINTER 18

You have three hours.

Problem	Points	Score
1	30	
2	10	
3	15	
4	10	
5	15	
6	10	
7	10	
8	15	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
Total	125	

There are 9 problems, and the total number of points is 125. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Student ID #:______________________________

1. (30pts) *Give the definition of*
(i) $\tau(n)$.

The number of divisors of n.

(ii) $\sigma(n)$.

The sum of the divisors of n.

(iii) $f(x) = O(g(x))$.

There is a constant $M > 0$ such that
$$\frac{|f(x)|}{g(x)} < M,$$
for x sufficiently large.

(iv) *Euler's constant.*

Euler's constant γ is the real number such that

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma n + O\left(\frac{1}{n}\right).$$

(v) completely multiplicative.

A function $f\colon \mathbb{N} \longrightarrow \mathbb{C}$ is completely multiplicative if

$$f(mn) = f(m)f(n),$$

for all m and n.

(vi) Twin primes.

Two primes p and q are twin primes if $q = p + 2$ (or vice-versa).

2. (10pts) (i) *Show that $\tau(n)$ is multiplicative.*

Suppose that m and n are coprime. Then d divides mn if and only if we may write $d = d_1 d_2$, where d_1 divides m and d_2 divides n. Thus the number of divisors of mn is the number of divisors of m multiplied by the number of divisors of n,

$$\tau(mn) = \tau(m)\tau(n).$$

(ii) *Find an expression for $\sigma_k(n)$ the sum of the kth powers of the divisors of n.*

Note that

$$\sigma_k(n) = \sum_{d|n} d^k.$$

As $m \longrightarrow m^k$ is multiplicative, it follows that σ_k is multiplicative. If $n = p^e$ is a power of a prime then

$$\begin{aligned}\sigma_k(n) &= \sum_{i=0}^{e} p^e \\ &= \frac{p^{e+1} - 1}{p - 1}.\end{aligned}$$

If $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ is the prime factorisation of n then

$$\sigma_k(n) = \prod_{i=1}^{k} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

3. (15pts) (i) *State the Möbius inversion formula.*
If

$$F(n) = \sum_{d|n} f(d) \qquad \text{then} \qquad f(n) = \sum_{d|n} \mu(d) F(n/d).$$

(ii) *Let*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n \text{ is a power of any prime } p, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\log n = \sum_{d|n} \Lambda(d)$$

Call a function $f \colon \mathbb{N} \longrightarrow \mathbb{N}$ logarithmic if $f(mn) = f(m) + f(n)$, whenever m and n are coprime. Note that both sides are logarithmic and so we are reduced to the case that $n = p^e$ is a power of a prime. In this case

$$\begin{aligned} \sum_{d|n} \Lambda(d) &= \sum_{i=0}^{e} \Lambda(p^i) \\ &= e \log p \\ &= \log p^e \\ &= \log n. \end{aligned}$$

(iii) *Show that*

$$\sum_{d|n} \mu(d) \log d = -\Lambda(n).$$

By Möbius inversion we have

$$\begin{aligned} \Lambda(n) &= \sum_{d|n} \mu(d) \log n/d \\ &= \sum_{d|n} \log n \mu(d) - \sum_{d|n} \mu(d) \log d \\ &= \log n (\sum_{d|n} \mu(d)) - \sum_{d|n} \mu(d) \log d \\ &= -\sum_{d|n} \mu(d) \log d. \end{aligned}$$

4. (10pts) *If n is a natural number, p is a prime and $n! = p^e m$, where m is a natural number coprime to p then prove that*

$$e = \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \dots.$$

Note that

$$e = \sum_{i=1}^{\infty} i e_i,$$

where e_i is the number of integers from 1 to n divisible by p^i but not p^{i+1}. Let f_i be the number of integers from 1 to n divisible by p^i. Note that $e_i = f_i - f_{i+1}$ so that

$$\begin{aligned} e &= \sum_{i=1}^{\infty} i e_i \\ &= \sum_{i=1}^{\infty} i(f_i - f_{i+1}) \\ &= \sum_{i=1}^{\infty} i f_i - \sum_{i=1}^{\infty} i f_{i+1} \\ &= \sum_{i=1}^{\infty} i f_i - \sum_{i=1}^{\infty} (i-1) f_i \\ &= \sum_{i=1}^{\infty} f_i. \end{aligned}$$

On the other hand, of the numbers from 1 to n, there are

$$f_i = \lfloor \frac{n}{p^i} \rfloor$$

numbers divisible by p^i.

5. (15pts) (i) *Let r be a natural number and let x be a real number. Show that*

$$\pi(x) \leq r + x\prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right) + 2^r.$$

Let $A(x,r)$ be the number of integers up to x divisible by the first r primes. Then

$$\pi(x) \leq r + A(x,r),$$

and by inclusion-exclusion

$$A(x,r) = \llcorner x \lrcorner - \sum_{i=1}^{r} \llcorner \frac{x}{p}_i \lrcorner + \sum_{i<j} \llcorner \frac{x}{p_i p_j} \lrcorner + \dots.$$

If we ignore the round down then we introduce an error of at most 2^r. Note that

$$x\prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right) = x\left(-\sum_{i=1}^{r}\frac{1}{p}_i + \sum_{i<j}\frac{1}{p_i p_j} + \dots\right)$$

$$= x - \sum_{i=1}^{r}\frac{x}{p}_i + \sum_{i<j}\frac{x}{p_i p_j} + \dots.$$

Putting all of this together we get

$$\pi(x) \leq r + x\prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right) + 2^r.$$

(ii) *If $x \geq 2$ then*

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right) < \frac{1}{\log x}.$$

We have

$$\left(1 - \frac{1}{p}\right)^{-1} = 1 + \frac{1}{p} + \frac{1}{p^2} + \dots.$$

As the RHS is absolutely convergent if multiply out all of these terms we get the sum of the reciprocral of every natural number divisible by only the first r primes. These include every number up to x.

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}=\sum_{k\le x}\frac{1}{k}.$$

If we view the last term as a Riemann sum then we get

$$\sum_{k\le x}\frac{1}{k}=\int_1^{\llcorner x\lrcorner+1}\frac{\mathrm{d}u}{u}>\log x.$$

(iii) *Show that*

$$\pi(x)\ll\frac{x}{\log\log x}.$$

By (i) and (ii)

$$\pi(x)\le 2^{r+1}+\frac{x}{\log p_r}.$$

As $p_r\ge r$ it follows that

$$\pi(x)\le 2^{r+1}+\frac{x}{\log r}.$$

Let $r=\log\llcorner x\lrcorner+1$. Then

$$\begin{aligned}\pi(x)&<\frac{x}{\log\log x}+4\cdot 2^{\log x}\\&<\frac{x}{\log\log x}+O(x^{\log 2}).\end{aligned}$$

As $\log 2<1$ it follows that

$$O(x^{\log 2})=o\left(\frac{x}{\log\log x}\right).$$

Thus

$$\pi(x)\ll\frac{x}{\log\log x}.$$

6. (10pts) *Let n be an integer and let p be a prime. If r_p is the unique integer such that*

$$p^{r_p} \leq 2n < p^{r_p+1},$$

then prove that

$$\prod_{n<p\leq 2n} p \,\Big|\, \frac{(2n)!}{n!n!} \qquad \text{and} \qquad \frac{(2n)!}{n!n!} \,\Big|\, \prod_{p\leq 2n} p^{r_p}.$$

If p is a prime and $p \leq 2n$ then p divides $(2n)!$. If $p > n$ then p does not divide $n!$. It follows that

$$\prod_{n<p\leq 2n} p \,\Big|\, \frac{(2n)!}{n!n!}.$$

Note that

$$\sum_{m=1}^{r_p} \left\lfloor \frac{2n}{p^m} \right\rfloor$$

is the exponent of p which divides $(2n)!$. On the other hand

$$2\sum_{m=1}^{r_p} \left\lfloor \frac{n}{p^m} \right\rfloor$$

is the exponent of p which divides $n!n!$. The difference

$$\begin{aligned}\sum_{m=1}^{r_p} \left\lfloor \frac{2n}{p^m} \right\rfloor - 2\sum_{m=1}^{r_p} \left\lfloor \frac{n}{p^m} \right\rfloor &= \sum_{m=1}^{r_p} \left\lfloor \frac{2n}{p^m} \right\rfloor - 2\left\lfloor \frac{n}{p^m} \right\rfloor \\ &\leq \sum_{m=1}^{r_p} 1 \\ &= r_p.\end{aligned}$$

is the exponent of p which divides

$$\frac{(2n)!}{n!n!}.$$

Thus

$$\frac{(2n)!}{n!n!} \,\Big|\, \prod_{p\leq 2n} p^{r_p}.$$

7. (10pts) *Show that if $s > 1$ then*

$$\prod_p \frac{1}{1-p^{-s}} = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

Call the RHS $\zeta(s)$.
If we expand the RHS for all of the primes up to x we get

$$\prod_{p \le x} \left(\frac{1}{1-p^{-s}} \right) = \prod_{p \le x} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots \right).$$

The product on the right is a finite product, over finitely many primes, of absolutely convergent geometric series. Thus we may rearrange the terms of the sum in any convenient order. If we expand the product we then get

$$\begin{aligned} \prod_{p \le x} \left(\frac{1}{1-p^{-s}} \right) &= \sum_{k: p|k \implies p \le x} \frac{1}{k^s} \\ &= \sum_{k \le x} \frac{1}{k^s} + \sum_{k > x: p|k \implies p \le x} \frac{1}{k^s} \\ &= \Sigma_1(x) + \Sigma_2(x). \end{aligned}$$

As the series

$$\sum_{k=1}^{\infty} \frac{1}{k^s}$$

converges, it follows that $\Sigma_1(x)$ converges to $\zeta(s)$ and $\Sigma_2(x)$ tends to zero.

8. (15pts) (i) *State Bertrand's hypothesis.*

If n is a natural number then there is a prime number p such that $n < p \leq 2n$.

(ii) *Show that every integer* $n > 6$ *is the sum of distinct primes (Hint: show that* every integer $6 < n \leq 19$ is a sum of distinct primes $p < 13$).

Note that every integer $7 \leq m \leq 19$ is a sum of distinct primes $p < 13$

$$7 = 7, \quad 8 = 5{+}3, \quad 9 = 7{+}2, \quad 10 = 7{+}3, \quad 11 = 11, \quad 12 = 7{+}5, \quad 13 = 11{+}2,$$
$$14 = 11{+}3, \quad 15 = 7{+}5{+}3, \quad 16 = 11{+}5, \quad 17 = 7{+}5{+}3{+}2, \quad 18 = 11{+}7, \quad 19 = 11{+}5{+}3.$$

Suppose that $19 < m \leq 26$. Then $7 \leq m - 13 \leq 13$. Thus $m - 13$ is a sum of primes less than 13 and adding 13, m is a sum of distinct primes. Thus every integer $7 \leq m \leq 26$ is a sum of distinct primes $p \leq 13$.
Now suppose that we know every integer $7 \leq m \leq 2p$ is a sum of distinct primes at most p. Pick a prime $p < q \leq 2p$. If $2p < m \leq 2q$ then $m - q \leq q \leq 2p$. In this case $m - q$ is a sum of distinct primes less than p and adding on q, m is a sum of distinct primes at most $q > p$.

9. (10pts) *Show that the number of lattice points inside the circle* $x^2 + y^2 \leq n$ is equal to

$$\pi n + O(\sqrt{n}).$$

The number of lattice points is

$$\sum_{|x| \leq \sqrt{n}} \sum_{|y| \leq \sqrt{n - x^2}} 1 = 2 \sum_{|x| \leq \sqrt{n}} \lfloor \sqrt{n - x^2} \rfloor$$

$$= 4 \sum_{0 \leq x \leq \sqrt{n}} \sqrt{n - x^2} + O(\sqrt{n}).$$

To estimate the last sum we use Riemann sums:

$$\sum_{0 < x \leq \sqrt{n}} \sqrt{n - x^2} \leq \int_0^n \sqrt{n - t^2}\, \mathrm{d}t \leq \sum_{0 \leq x < \sqrt{n}} \sqrt{n - x^2}.$$

As

$$\int_0^n \sqrt{n - t^2}\, \mathrm{d}t = \pi n$$

and the difference between the upper and low sum is bounded by $\sqrt{n}$, the number of lattice points inside the circle $x^2 + y^2 \leq n$ is

$$\pi n + O(\sqrt{n}).$$

Bonus Challenge Problems
11. (10pts) *State and prove the formula for partial summation.*

Suppose that $\lambda_1, \lambda_2, \ldots$ is a sequence of reals such that

$$\lambda_1 \leq \lambda_2 \leq \ldots$$

and the limit is infinity. Let $c_1, c_2, \ldots$ be any sequence of complex numbers and let $f(x)$ be a function whose derivative is continuous for $x \geq \lambda_1$.
If

$$C(x) = \sum_{\lambda_n \leq x} c_n,$$

where the sum is over all n such that $\lambda_n \leq x$, then

$$\sum_{\lambda_n \leq x} c_n f(\lambda_n) = C(x)f(x) - \int_{\lambda_1}^{x} C(t)f'(t)\,\mathrm{d}t.$$

If ν is the largest index such that $\lambda_\nu \leq x$, we have

$$\begin{aligned}\sum_{\lambda_n \leq x} c_n f(\lambda_n) &= C(\lambda_1)f(\lambda_1) + (C(\lambda_2) - C(\lambda_1))f(\lambda_2) + \cdots + (C(\lambda_\nu) - C(\lambda_{\nu-1}))f(\lambda_n) \\ &= C(\lambda_1)(f(\lambda_1) - f(\lambda_2)) + \cdots + C(\lambda_{\nu-1})(f(\lambda_\nu) - f(x)) + C(\lambda_{\lambda_\nu})f(x) \\ &= -\int_{\lambda_1}^{x} C(t)f'(t)\,\mathrm{d}t + C(x)f(x),\end{aligned}$$

since $C(t)$ is constant over the intervals $(\lambda_{i-1}, \lambda_i)$ and (λ_ν, x).

11. (10pts) *Show that*

$$\sum_{m=1}^{n} \tau(m) = n \log n + (2\gamma - 1)n + O(\sqrt{n}).$$

We have

$$\begin{aligned}\sum_{m=1}^{n} \tau(m) &= \sum_{m=1}^{n} \sum_{d|m} 1 \\ &= \sum_{d=1}^{n} \sum_{m=1}^{n/d} 1 \\ &= \sum_{d=1}^{n} \llcorner \frac{n}{d} \lrcorner.\end{aligned}$$

Geometrically, the sum on the RHS is the number of lattice points (x, y) where x and y are natural numbers, on or below the hyperbola $xy = n$, since if we fix x the number of $1 \leq y \leq n/x$ is precisely $\llcorner n/x \lrcorner$. By symmetry the number of lattice points (x, y) with $x > 0$, $y > 0$ and $xy \leq n$ is equal to twice the number of lattice points $x > 0$, $y > x$ and $xy \leq n$ plus the number of lattice points $x > 0$, $x = y$ and $xy \leq n$. Hence

$$\begin{aligned}\sum_{m=1}^{n} \tau(m) &= 2\left(\sum_{x=1}^{\sqrt{n}} \llcorner \frac{n}{x} \lrcorner - x\right) + \llcorner \sqrt{n} \lrcorner \\ &= 2n\left(\sum_{x=1}^{\sqrt{n}} \frac{1}{x}\right) + O(\sqrt{n}) - 2\frac{\llcorner \sqrt{n} \lrcorner(\llcorner \sqrt{n} \lrcorner + 1)}{2} \\ &= 2n(\log(\sqrt{n}) + \gamma + O(1/\sqrt{n})) - n + O(\sqrt{n}) \\ &= n \log n + (2\gamma - 1)n + O(\sqrt{n}).\end{aligned}$$

12. (10pts) *Show that*

$$\sum_{n\leq x}\frac{\tau(n)}{n}=\frac{1}{2}\log^2 x+2\gamma x+O(1).$$

We apply partial summation to

$$\lambda_n=n \qquad c_n=\tau(n) \qquad \text{and} \qquad f(x)=\frac{1}{x}.$$

We get

$$\sum_{n\leq x}\frac{\tau(n)}{n}=\frac{\sum_{n\leq x}\tau(n)}{x}+\int_1^x\frac{\sum_{n\leq t}\tau(n)}{t^2}\,\mathrm{d}t.$$

Now

$$\sum_{n\leq x}\tau(n)=x\log x+(2\gamma-1)x+O(x^{1/2}).$$

Thus

$$\begin{aligned}\int_1^x\frac{\sum_{n\leq t}\tau(n)}{t^2}\,\mathrm{d}t&=\int_1^x\frac{\log t}{t}\,\mathrm{d}t+(2\gamma-1)\int_1^x\frac{1}{t}\,\mathrm{d}t+\int_1^x\frac{O(t^{1/2})}{t^2}\,\mathrm{d}t\\&=\left[\frac{1}{2}\log^2 t\right]_1^x+(2\gamma-1)\Big[\log t\Big]_1^x+O\left(\int_1^\infty\frac{1}{t^{3/2}}\,\mathrm{d}t\right)\\&=\frac{1}{2}\log^2 x+(2\gamma-1)\log x+O(1).\end{aligned}$$

Hence

$$\sum_{n\leq x}\frac{\tau(n)}{n}=\frac{1}{2}\log^2 x+2\gamma\log x+O(1).$$

13. (10pts) *Show that*

$$\lim_{s \to 1^+} \prod_q (1 - q^{-s})^{-1} = \infty,$$

where the product ranges over the primes q congruent to one modulo four.

See lecture 9.

14. (10pts) *Prove Brun's theorem.*

See lecture 15.