12. Bertrand’s Hypothesis

We will prove:

**Theorem 12.1** (Bertrand’s hypothesis). *If $n$ is a natural number then there is a prime $p$ such that $n < p \leq 2n$.***

Bertrand’s hypothesis was first formulated by Bertrand, who checked it up to six million. He stated it as a hypothesis rather than a conjecture, as he was very confident it was correct. For example, between 500,000 and 1,000,000 there are 36,960 primes.

(12.1) was proved by Chebyshev. In fact he also proved that for every $\epsilon > 1/5$ there is a number $\xi$ such that for all $x > \xi$ we may find a prime $p$ such that $x \leq p \leq (1 + \epsilon)x$ (in fact the prime number theorem implies that this is true for any $\epsilon > 0$).

**Lemma 12.2.**

$$\prod_{p \leq n} p < 4^n,$$

for every natural number $n$.

**Proof.** By induction on $n$. The cases $n = 1$ and $n = 2$ are clear.

Now suppose it is true up to $n - 1$, where $n \geq 3$. If $n$ is even then

$$\prod_{p \leq n} p = \prod_{p \leq n-1} p \cdot p < 4^{n-1} < 4^n.$$ 

Thus we may assume that $n = 2m + 1$ is odd.

Consider the binomial coefficient

$$\binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!}.$$ 

This is divisible by every prime $m + 2 \leq p \leq 2m + 1$. Thus

$$\prod_{p \leq 2m+1} p \leq \binom{2m+1}{m} \cdot \prod_{p \leq m+1} p < \binom{2m+1}{m} \cdot 4^{m+1}.$$ 

On the other hand,

$$\binom{2m+1}{m} \quad \text{and} \quad \binom{2m+1}{m+1}$$

**
are equal and both appear in the binomial expansion of \((1 + 1)^{2m+1}\). Therefore
\[
\binom{2m + 1}{m} \leq \frac{1}{2} 2^{2m+1} = 4^m.
\]
Hence
\[
\prod_{p \leq 2m+1} p < 4^m \cdot 4^{m+1} = 4^{2m+1}.
\]
This completes the induction and the proof. \(\square\)

**Lemma 12.3.** If \(n \geq 3\) and \(2n/3 < p \leq n\) then \(p\) does not divide \(\binom{2n}{n}\).

**Proof.** Note that
(1) \(p\) is greater than 2.
(2) \(p\) and \(2p\) are the only multiples of \(p\) less than or equal to \(2n\), since \(3p > 2n\).
(3) \(p\) is at most \(n\).

(1) and (2) imply \((2n)!\) is divisible by \(p^2\) but not \(p^3\) and (3) implies that \(p^2\) divides \((n!)^2\). Thus \(p\) does not divide \(\binom{2n}{n} = \frac{(2n)!}{(n!)^2}\). \(\square\)

**Proof of [12.1].** We can do the cases \(n = 1\) and \(n = 2\) by hand.

Suppose there is no prime between \(n\) and \(2n\), where \(n \geq 3\). We will bound \(n\) from above. [12.3] implies that if \(p\) divides \(\binom{2n}{n}\) then \(p \leq 2n/3\). Suppose that
\[
\binom{2n}{n} = p^r m,
\]
where \(m\) is coprime to \(p\). By the proof of (7.1)
\[
\binom{2n}{n} \bigg| \prod_{p \leq 2n} p^{r_p}
\]
where \(r_p\) is the unique integer such that
\[p^{r_p} \leq 2n < p^{r_p+1}.
\]
Thus \(p^r \leq 2n\).
If \( e \geq 2 \) then \( p \leq \sqrt{2n} \). In particular there are at most \( \sqrt{2n} \) primes in the prime factorisation of

\[
\binom{2n}{n}
\]

with exponent greater than 1. Therefore

\[
\binom{2n}{n} \leq (2n)^{\sqrt{2n}} \cdot \prod_{p \leq 2n/3} p.
\]

On the other hand

\[
\binom{2n}{n}
\]

is the largest of the \( 2n + 1 \) terms in the expansion of \((1 + 1)^{2n}\), so that

\[
4^n < (2n + 1)\binom{2n}{n}.
\]

It follows that

\[
4^n < (2n + 1)(2n)^{\sqrt{2n}} \cdot \prod_{p \leq 2n/3} p.
\]

(12.2) implies that

\[
4^n < (2n + 1)(2n)^{\sqrt{2n}} \cdot 4^{2n/3}.
\]

As \( 2n + 1 < 4n^2 \) we get

\[
4^n < (2n)^{\sqrt{2n} + 2} \cdot 4^{2n/3},
\]

so that

\[
4^{n/3} < (2n)^{\sqrt{2n} + 2}.
\]

Taking logs gives

\[
\frac{n \log 4}{3} < (\sqrt{2n} + 2) \log(2n).
\]

Note that

\[
512 = 2^9.
\]

If we plug in \( n = 512 \) to the equation above the RHS is

\[
(2^5 + 2)10 \log 2 = 340 \log 2
\]

and the LHS is

\[
\frac{2^{10} \log 2}{3} > 341 \log 2.
\]

Thus \( n < 512 \). Thus if \( n \geq 512 \) there is a prime between \( n \) and \( 2n \).

On the other hand, for the sequence of primes

\[
2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 557,
\]
each prime is smaller than twice the preceding prime, so that there is a prime between $n$ and $2n$ for $n < 512$ as well.  
\[ \Box \]