6. Sums involving primes

**Lemma 6.1.** If $t > -1$ is a non-zero real number then

$$
\frac{t}{1 + t} < \log(1 + t) < t.
$$

In particular if $t \in (0, 1/2]$ then

$$
\log(1 - t) > -2t.
$$

**Proof.** Let

$$
f: (-1, \infty) \rightarrow \mathbb{R}
$$

be the function

$$
f(u) = \log(1 + u).
$$

Then $f$ is continuous and differentiable, so that by the mean value theorem we may find

$$
s \in \begin{cases}
[0, t] & \text{if } t > 0 \\
t, 0 & \text{if } t < 0
\end{cases}
$$

such that

$$
\frac{1}{1 + s} = f'(s) = \frac{f(t) - f(0)}{t - 0} = \frac{\log(1 + t) - \log 1}{t - 0} = \frac{\log(1 + t)}{t}.
$$

It follows that

$$
\log(1 + t) = \frac{t}{1 + s}.
$$

Now use the fact that the RHS is a decreasing function of $s$, if $t > 0$ and an increasing function of $s$, if $t < 0$ to get the first inequality.

Now suppose that $t \in (0, 1/2]$, so that $-t \in [-1/2, 0)$. As $-t \geq -1/2$, $1 - t \geq 1/2$ and so

$$
\log(1 - t) > \frac{-t}{1 - t} \geq -2t.
$$

\[\square\]

**Theorem 6.2.**

$$
\sum_{p \leq x} \frac{1}{p} > \frac{1}{2} \log \log x.
$$
In particular
\[ \sum_p \frac{1}{p} \]
diverges.

Proof. We start with the inequality of (5.1),
\[ \prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{1}{\log x}. \]
If we take logs of both sides, we get
\[ \sum_{p \leq x} \log \left(1 - \frac{1}{p}\right) = \log \prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \log \frac{1}{\log x} = -\log \log x. \]
Now (6.1) implies that
\[ \left(1 - \frac{1}{p}\right) > \frac{-2}{p} \]
for every prime \( p \) and so
\[ \sum_{p \leq x} \frac{2}{p} > \log \log x. \]
\[ \square \]

**Theorem 6.3.**
\[ \sum_{p \leq x} \frac{\log p}{p} = \log x + O\left(\frac{\log x}{\log \log x}\right). \]

Proof. We saw in (3.3) that if we write \( n = p^m \), where \( m \) is coprime to \( p \), then
\[ r = \frac{n}{p} + \frac{n^2}{p^2} + \frac{n^3}{p^3} + \ldots. \]
It follows that
\[ n! = \prod_{p \leq n} p^{r - \frac{n}{p} + \frac{n^2}{p^2} + \frac{n^3}{p^3} + \ldots}, \]
so that
\[ \log n! = \sum_{p \leq n} (\frac{n}{p} - \frac{n^2}{p^2} + \frac{n^3}{p^3} + \ldots) \log p. \]
For the first term in every sum on the RHS we have
\[ \sum_{p \leq n} \frac{n}{p} \log p \leq \sum_{p \leq n} \frac{n}{p} \log p. \]
On the other hand,
\[
\sum_{p \leq n} \frac{n}{p} \log p \geq \sum_{p \leq n} \left( \frac{n}{p} - 1 \right) \log p
\]
\[
= \sum_{p \leq n} \frac{n}{p} \log p - \sum_{p \leq n} \log p
\]
\[
\geq \sum_{p \leq n} \frac{n}{p} \log p - \log n \sum_{p \leq n} 1.
\]
\[
= \sum_{p \leq n} \frac{n}{p} \log p - \pi(n) \log n.
\]

Note that
\[
\sum_{p \leq n} \left( \frac{n}{p^2} + \frac{n}{p^3} + \ldots \right) \log p \leq \sum_{p \leq n} \left( \frac{n}{p^2} + \frac{n}{p^3} + \ldots \right) \log p.
\]

Thus
\[
\log n! = \sum_{p \leq x} \frac{n}{p} \log p + O\left( \pi(n) \log n \right) + O\left( n \sum_{p \leq n} \left( \frac{1}{p^2} + \frac{1}{p^3} + \ldots \right) \log p \right)
\]
\[
= n \sum_{p \leq x} \frac{\log p}{p} + O\left( \pi(n) \log n \right) + O\left( n \sum_{p \leq n} \frac{\log p}{p(p-1)} \right).
\]

Since the series
\[
\sum_{k=1}^{\infty} \frac{\log k}{k(k-1)}
\]
converges, we get
\[
\log n! = n \sum_{p \leq x} \frac{\log p}{p} + O\left( \pi(n) \log n \right) + O(n).
\]

However it is shown in a homework problem that
\[
\log n! = n \log n + O(n).
\]

Putting all of this together we get
\[
n \sum_{p \leq x} \frac{\log p}{p} = n \log n + O(n) + O(\pi(n) \log n).
\]

But by (5.2)
\[
\pi(n) = O\left( \frac{n}{\log \log n} \right).
\]
Dividing through by $n$ we get
\[
\sum_{p \leq x} \frac{\log p}{p} = \log n + O\left(\frac{\log n}{\log \log n}\right).
\]
This establishes the result when $x = n$ is an integer. As
\[
\log x = \log \lfloor x \rfloor + O(1),
\]
the result holds for any $x$. \qed

We will need the following result, which is obtained using integration by parts.

**Theorem 6.4 (Summation formula).** Suppose that $\lambda_1, \lambda_2, \ldots$ is a sequence of reals such that
\[
\lambda_1 \leq \lambda_2 \leq \ldots
\]
and the limit is infinity. Let $c_1, c_2, \ldots$ be any sequence of complex numbers and let $f(x)$ be a function whose derivative is continuous for $x \geq \lambda_1$.

If\[
C(x) = \sum_{\lambda_n \leq x} c_n,
\]
where the sum is over all $n$ such that $\lambda_n \leq x$, then
\[
\sum_{\lambda_n \leq x} c_n f(\lambda_n) = C(x)f(x) - \int_{\lambda_1}^{x} C(t)f'(t) \, dt.
\]

**Proof.** If $\nu$ is the largest index such that $\lambda_\nu \leq x$, we have
\[
\sum_{\lambda_n \leq x} c_n f(\lambda_n) = C(\lambda_1)f(\lambda_1) + (C(\lambda_2) - C(\lambda_1))f(\lambda_2) + \cdots + (C(\lambda_\nu) - C(\lambda_{\nu-1}))f(\lambda_\nu)
\]
\[
= C(\lambda_1)(f(\lambda_1) - f(\lambda_2)) + \cdots + C(\lambda_{\nu-1})(f(\lambda_{\nu-1}) - f(x)) + C(\lambda_\nu)f(x)
\]
\[
= -\int_{\lambda_1}^{x} C(t)f'(t) \, dt + C(x)f(x),
\]
since $C(t)$ is constant over the intervals $(\lambda_{i-1}, \lambda_i)$ and $(\lambda_\nu, x)$. \qed

**Theorem 6.5.**
\[
\sum_{p \leq x} \frac{1}{p} \sim \log \log x.
\]

**Proof.** We are going to apply (6.4) to
\[
\lambda_n = p_n, \quad c_n = \frac{\log p_n}{p_n}, \quad \text{and} \quad f(t) = \frac{1}{\log t}.
\]
We have
\[
\sum_{p \leq x} \frac{1}{p} = \sum_{2 < p < x} \left( \frac{\log p}{p} \cdot \frac{1}{\log p} \right) + \frac{1}{2}
\]
\[
= \frac{1}{\log x} \sum_{2 < p < x} \frac{\log p}{p} - \int_{3}^{x} \left( \sum_{p \leq t} \frac{\log p}{p} \right) \frac{-dt}{t \log^2 t} + O(1)
\]
\[
= \frac{1}{\log x} \left( \log x + O \left( \frac{\log x}{\log \log x} \right) \right) + \int_{3}^{x} \left( \log t + O \left( \frac{\log t}{\log \log t} \right) \right) \frac{dt}{t \log^2 t} + O(1)
\]
\[
= \int_{3}^{x} \frac{dt}{t \log t} + \int_{3}^{x} O \left( \frac{1}{t \log t \log \log t} \right) \ dt + O(1)
\]
\[
= \log \log x + \int_{3}^{x} O \left( \frac{1}{t \log t \log \log t} \right) \ dt + O(1)
\]

For the last term there is some constant $M$ such that
\[
\left| \int_{3}^{x} O \left( \frac{1}{t \log t \log \log t} \right) \ dt \right| \leq M \int_{3}^{x} \frac{dt}{t \log t \log \log t}
\]
\[
= M \log \log \log x + O(1).
\]

It follows that
\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + O \left( \log \log \log x \right). \quad \square
\]