FIRST MIDTERM
MATH 104B, UCSD, WINTER 18

You have 80 minutes.

There are 4 problems, and the total number of points is 70. Show all your work. Please make your work as clear and easy to follow as possible.

<table>
<thead>
<tr>
<th>Name:</th>
<th>Signature:</th>
<th>Student ID #:</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>70</td>
<td></td>
</tr>
</tbody>
</table>
1. (15pts) (i) *Give the definition of $\pi(x)$.\*

The number of primes up to $x$.

(ii) *Give the definition of the Möbius function.*

The function $\mu: \mathbb{N} \rightarrow \mathbb{N}$ defined by the rule

$$
\mu(n) = \begin{cases} 
(-1)^\nu & \text{if } n \text{ is the product of } \nu \text{ distinct primes} \\
0 & \text{otherwise.}
\end{cases}
$$

(iii) *Give the definition of the fractional part.*

If $x$ is real number and $\lfloor x \rfloor$ is the largest integer less than $x$ then the fractional part is

$$\{ x \} = x - \lfloor x \rfloor$$
2. (15pts) Let $x, x_1$ and $x_2$ be real numbers and let $n$ be an integer. Prove that

(i) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

As $\lfloor x \rfloor \leq x$ it follows that $\lfloor x \rfloor + n \leq \lfloor x + n \rfloor$. As $\lfloor x + n \rfloor \leq x + n$ it follows that $\lfloor x + n \rfloor - n \leq x$. As the LHS is an integer it follows that $\lfloor x + n \rfloor - n \leq \lfloor x \rfloor$. Adding $n$ to both sides we get $\lfloor x + n \rfloor \leq \lfloor x \rfloor + n$. As we have an inequality both ways we must have an equality $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

(ii) $\lfloor x_1 \rfloor + \lfloor x_2 \rfloor \leq \lfloor x_1 + x_2 \rfloor$.

Since $\lfloor x_i \rfloor \leq x_i$ for $i = 1, 2$ we have $\lfloor x_1 \rfloor + \lfloor x_2 \rfloor \leq x_1 + x_2$. As the LHS is an integer it follows that $\lfloor x_1 \rfloor + \lfloor x_2 \rfloor \leq \lfloor x_1 + x_2 \rfloor$.

(iii) Assuming that $n$ is a natural number, prove that

$$\frac{x}{n} = \frac{\lfloor x \rfloor}{n}.$$ 

As $\lfloor x \rfloor \leq x$ it follows that $\frac{\lfloor x \rfloor}{n} \leq \frac{x}{n}$.

But then $\frac{\lfloor x \rfloor}{n} \leq \frac{x}{n}$ so that $\frac{\lfloor x \rfloor}{n} \leq \frac{\lfloor x \rfloor}{n}$, as the LHS is an integer.

On the other hand, as $\frac{x}{n} \leq \frac{\lfloor x \rfloor}{n}$ it follows that $n\frac{x}{n} \leq x$.

so that $\frac{x}{n} \leq \frac{\lfloor x \rfloor}{n}$ and so $\frac{x}{n} \leq \frac{\lfloor x \rfloor}{n}$.

As we have an inequality both ways, we have equality.
3. (30pts) Let \( f : \mathbb{N} \rightarrow \mathbb{C} \) be a function and define \( F : \mathbb{N} \rightarrow \mathbb{C} \) by the rule
\[
F(n) = \sum_{d|n} f(d).
\]

(i) Show that if \( f \) is multiplicative then \( F \) is multiplicative.

Suppose that \( m \) and \( n \) are coprime. Note that if \( d \) divides \( mn \) then \( d = d_1d_2 \) where \( d_1 \) divides \( m \) and \( d_2 \) divides \( n \). We have
\[
F(mn) = \sum_{d|m,n} f(d)
\]
\[
= \sum_{d_1|m, d_2|n} f(d_1d_2)
\]
\[
= \sum_{d_1|m} \sum_{d_2|n} d_2|n f(d_1)f(d_2)
\]
\[
= \sum_{d_1|m} f(d_1) \sum_{d_2|n} f(d_2)
\]
\[
= F(m)F(n).
\]

Thus \( F \) is multiplicative.
(ii) If the function

\[
M : \mathbb{N} \rightarrow \mathbb{Z}
\]

is defined by \( M(n) = \sum_{d|n} \mu(d) \)

then show that

\[
M(n) = \begin{cases} 
1 & \text{if } n = 1 \\
0 & \text{otherwise.} 
\end{cases}
\]

Consider

\[
M(n) = \sum_{d|n} \mu(d). 
\]

As \( \mu(n) \) is multiplicative both sides are multiplicative. If \( n = p^e \) is a power of a prime then

\[
M(p^e) = \mu(1) + \mu(p) + \mu(p^2) + \cdots = 1 - 1 + 0 + \cdots + 0 = 0.
\]

Thus \( M(n) = 0 \) unless \( n = 1 \) in which case \( M(1) = 1 \).

(iii) Show that

\[
f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right). 
\]

We have

\[
\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d_1 d_2 = n} \mu(d_1) F(d_2)
\]

\[
= \sum_{d_1 d_2 = n} \mu(d_1) \sum_{d|d_2} f(d)
\]

\[
= \sum_{d_1|d|n} \mu(d_1) f(d)
\]

\[
= \sum_{d|n} f(d) \sum_{d_1|n/d} \mu(d_1)
\]

\[
= \sum_{d|n} f(d) M(n/d)
\]

\[
= f(n).
\]
4. (10pts) Show that
\[ \pi(n) \geq \frac{\log n}{2 \log 2}. \]

Let \( r = \pi(n) \) and let \( p_1, p_2, \ldots, p_r \) be the first \( r \) primes, so that \( p_1, p_2, \ldots, p_r \) are the primes up to \( n \). Note that we may form \( 2^r \) distinct square-free natural numbers \( m \) which are only divisible by \( p_1, p_2, \ldots, p_r \). For each prime \( p_i \) we either choose \( m \) coprime to \( p_i \) or divisible by \( p_i \).

On other hand there are at most \( \sqrt{n} \) perfect squares up \( n \).

Now any natural number \( l \) is the product of a perfect square and a square-free number \( m \). If \( l \leq n \) then \( m \leq n \) and so \( m \) is divisible by \( p_1, p_2, \ldots, p_r \). Thus there are at most \( 2^r \sqrt{n} \) numbers up to \( n \). On the other hand there are \( n \) natural numbers up to \( n \).

Thus
\[ 2^{\pi(n)} \sqrt{n} \geq n \]

so that
\[ 2^{\pi(n)} \geq \sqrt{n}. \]

Taking logs we see that
\[
\begin{align*}
\pi(n) \log 2 &= \log 2^{\pi(n)} \\
&\geq \log \sqrt{n} \\
&= \frac{1}{2} \log n.
\end{align*}
\]

Thus
\[ \pi(n) \geq \frac{\log n}{2 \log 2}. \]
5. Show that
\[ \sum_{i<j=1}^{\infty} \frac{x}{p_i p_j} = x \sum_{p_i p_j \leq x, p_i < p_j} \frac{1}{p_i p_j} + O(x). \]

Note that
\[ \sum_{i,j=1; p_i < p_j}^{\infty} \frac{x}{p_i p_j} = \sum_{p_i p_j \leq x, p_i < p_j} \frac{x}{p_i p_j} \]
\[ \leq \sum_{p_i p_j \leq x, p_i < p_j} \frac{x}{p_i p_j} - E, \]
where
\[ E \leq \sum_{p_i p_j \leq x, p_i < p_j} 1. \]

The term on the RHS is the number of ways to pick two primes \( p_i, p_j \) such that \( p_i < p_j \) and \( p_i p_j \leq x \). Let \( y = p_i p_j \). Then \( y \) determines \( p_i \) and \( p_j \) by unique factorisation and \( y \) is a natural number between 1 and \( x \) so that \( y \leq \ll x \rr \leq x \).

Thus \( E \) is at most \( x \) and so \(-E = O(1)\).
Bonus Challenge Problems

6. (10pts) Show that there is a constant $\gamma$ such that

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + O\left(\frac{1}{n}\right).$$

Let

$$\alpha_k = \log k - \log(k-1) - \frac{1}{k}$$

and

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \log n.$$

Note that

$$1 - \gamma_n = \sum_{k=2}^{n} \alpha_k.$$

Note also that

$$\int_{k-1}^{k} \frac{1}{x} \, dx = [\log x]_{k-1}^{k}$$

$$= \log k - \log(k-1)$$

is the area under the curve $y = 1/x$ over the interval $k - 1 \leq x \leq k$. On the other hand $1/k$ is the area over the interval $k - 1 \leq x \leq k$ inside the largest rectangle inscribed between the $x$-axis and the curve $y = 1/x$.

It follows that $\alpha_k$ is the difference between these two areas, so that $\alpha_k$ is positive. Note that if we drop these areas down to the region between $x = 0$ and $x = 1$ then all of these areas fit into the unit square bounded by $y = 0$ and $y = 1$.

Thus $0 < 1 - \gamma_n < 1$ is bounded and monotonic increasing. It follows that $1 - \gamma_n$ tends to a limit. Define $\gamma$ by the formula:

$$\lim_{n \to \infty} (1 - \gamma_n) = 1 - \gamma.$$

Finally note that the difference

$$\gamma_n - \gamma = (1 - \gamma) - (1 - \gamma_n)$$

$$= \sum_{k=n+1}^{\infty} \alpha_k$$

is represented by an area which fits inside a box with one side 1 and the other side $1/n$, so that it is less than $1/n$. Thus

$$\gamma_n - \gamma = O\left(\frac{1}{n}\right).$$
7. (10pts) Show that

\[ \pi(x) = O\left(\frac{x}{\log \log x}\right). \]

The number of integers up to \( x \) not divisible by the first \( r \) primes \( p_1, p_2, \ldots, p_r \) is

\[ A(x, r) = \lfloor x \rfloor - \sum_{i=1}^{r} \frac{x}{p_i} + \sum_{i \neq j \leq r} \frac{x}{p_ip_j} + \cdots + (-1)^r \lfloor \frac{x}{p_1p_2 \ldots p_r} \rfloor. \]

by inclusion-exclusion. If we approximate this by ignoring the round downs the error is at most

\[ 1 + \binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{r} = 2^r, \]

and so

\[ \pi(x) \leq r + x \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) + 2^r. \]

Now

\[ \prod_{p \leq x} \frac{1}{1 - \frac{1}{p}} = \prod_{p \leq x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right). \]

If we expand the RHS we get the sum of the reciprocals of all numbers divisible by \( p_1, p_2, \ldots, p_r \), which is at least

\[ \sum_{k \leq n} \frac{1}{k} > \log n. \]

Thus

\[ \pi(x) \leq r + \frac{x}{\log x} + 2^r. \]

If we take \( r = \lceil \log x \rceil \) then

\[ \pi(x) \leq 2^{\log x + 2} + \frac{x}{\log \log x} \quad \text{take } r = \lceil \log x \rceil \]

\[ = O\left(2^{\log x}\right) + \frac{x}{\log \log x} \]

\[ \leq o\left(\frac{x}{\log \log x}\right) + \frac{x}{\log \log x} \quad \text{as } \log 2 < 1 \]

\[ = O\left(\frac{x}{\log \log x}\right). \]