SECOND MIDTERM
MATH 104B, UCSD, WINTER 18

You have 80 minutes.

There are 5 problems, and the total number of points is 70. Show all your work. *Please make your work as clear and easy to follow as possible.*

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Name:______________________________

Signature:_________________________

Student ID #:______________________
1. (15pts) (i) \textit{Give the definition of li}(x).

\[ \text{li}(x) = \int_2^x \frac{1}{\log t} \, dt. \]

(ii) \textit{Give the definition of } \vartheta(x).

\[ \vartheta(x) = \sum_{p \leq x} \log p. \]

(iii) \textit{Give the definition of the Riemann zeta-function } \zeta(s).

\[ \zeta(s) = \sum_k \frac{1}{k^s}. \]
2. (10pts) Show that

\[
\text{li}(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \frac{2!x}{\log^3 x} + \cdots + \frac{(n-1)!x}{\log^n x} + O\left(\frac{x}{\log^{n+1} x}\right).
\]

Note that if we integrate by parts then we get

\[
\int_0^x \frac{dt}{\log^n t} = \int_0^x \frac{1}{n} \cdot \frac{dt}{\log^n t} = \left[ \frac{t}{n \log^n t} \right]_0^x + n \int_0^x \frac{t}{t \log^{n+1} t} dt = \frac{x}{\log^n x} + n \int_0^x \frac{dt}{\log^{n+1} t}.
\]

It follows by induction that

\[
\text{li}(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \frac{2!x}{\log^3 x} + \cdots + \frac{(n-1)!x}{\log^n x} + n \int_0^x \frac{dt}{\log^{n+1} t}.
\]

Now to estimate the last integral, we break it into three parts.

\[
\int_0^x \frac{dt}{\log^{n+1} t} = \int_0^2 \frac{dt}{\log^{n+1} t} + \int_2^{\sqrt{x}} \frac{dt}{\log^{n+1} t} + \int_{\sqrt{x}}^x \frac{dt}{\log^{n+1} t}.
\]

The first integral is constant. The second is over an interval of length bounded by \(\sqrt{x}\) of a function bounded by a constant \(\frac{1}{\log^{n+1/2}}\) and so the second integral is \(O(\sqrt{x})\). The third integral is over an interval of length bounded by \(x\) of a function which is bounded by

\[
\frac{1}{\log^{n+1} \sqrt{x}} = O \left(\frac{1}{\log^{n+1} x}\right).
\]

Thus the last integral is

\[
O \left(\frac{x}{\log^{n+1} x}\right).
\]

Therefore

\[
\left| n! \int_0^x \frac{dt}{\log^{n+1} t} \right| = O \left(\frac{x}{\log^{n+1} x}\right).
\]

and so the result follows.
3. (15pts) Show that

\[
\pi(x) \leq r + x \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) + 2^r,
\]

where \(p_1, p_2, \ldots, p_r\) are the first \(r\) primes.

Let

\[P = \{ n \in \mathbb{N} \mid 1 < n \leq x \text{ and } n \text{ is a not a multiple of } p_1, p_2, \ldots, p_r \},\]

so that \(P\) is the set of integers from 2 to \(x\) which are not multiples of \(p_1, p_2, \ldots, p_r\). Let \(A(x, r)\) be the cardinality of \(P\).

If \(p\) is a prime from 1 to \(n\) then either \(p\) is one of \(p_1, p_2, \ldots, p_r\) or \(p\) belongs to \(P\). It follows that

\[
\pi(x) \leq r + A(x, r).
\]

We want to estimate \(A(x, r)\). Let \(M_i\) be the set of integers from 1 to \(n\) which are multiples of \(p_i\). Let \(M_{ij}\) be the set of integers from 1 to \(n\) which are multiples of both \(p_i\) and \(p_j\). As \(p_i\) and \(p_j\) are coprime,

\[M_{ij} = M_i \cap M_j.\]

Note that

\[|M_i| = \lfloor \frac{x}{p_i} \rfloor \quad \text{ and } \quad |M_{ij}| = \lfloor \frac{x}{p_ip_j} \rfloor,\]

and so on. It follows by inclusion-exclusion that

\[A(x, r) = \lfloor x \rfloor - \sum_{i=1}^{r} \lfloor \frac{x}{p_i} \rfloor + \sum_{i < j \leq r} \lfloor \frac{x}{p_ip_j} \rfloor + \cdots + (-1)^r \lfloor \frac{x}{p_1p_2 \cdots p_r} \rfloor.\]

Suppose that we approximate the RHS by simply ignoring all of the round downs,

\[x - \sum_{i=1}^{r} \frac{x}{p_i} + \sum_{i < j \leq r} \frac{x}{p_ip_j} + \cdots + (-1)^r \frac{x}{p_1p_2 \cdots p_r}.\]

The worse case scenario for the error is

\[1 + \binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{r} = 2^r.\]

It follows that

\[
\pi(x) \leq r + x \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) + 2^r.
\]
(b) If \( x \geq 2 \) then
\[
\prod_{p \leq x} \left( 1 - \frac{1}{p} \right) < \frac{1}{\log x}.
\]

We compute the product of the reciprocals,
\[
\prod_{p \leq x} \frac{1}{1 - \frac{1}{p}} = \prod_{p \leq x} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots \right).
\]
Consider what happens if we expand the RHS. If \( m \) is an integer which is a product of primes less than \( x \) then the term \( \frac{1}{m} \) appears somewhere in the expansion of this product.
Now any integer \( m \leq x \) is a product of primes less than \( x \) and so
\[
\prod_{p \leq x} \frac{1}{1 - \frac{1}{p}} > \sum_{k=1}^{n} \frac{1}{k} > \int_{1}^{\lfloor x \rfloor} \frac{du}{u} > \log x.
\]
\[ \pi(x) \leq \frac{x}{\log \log x} . \]

\[ \pi(x) \leq r + x \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right) + 2^r \quad \text{as proved above} \]

\[ \leq 2^{r+1} + x \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right) \quad \text{as } r \leq 2^r \]

\[ \leq 2^{r+1} + \frac{x}{\log p_r} \quad \text{using (b)} \]

\[ \leq 2^{r+1} + \frac{x}{\log r} \quad \text{as } p_r \geq r \]

\[ \leq 2^{\log x + 2} + \frac{x}{\log \log x} \quad \text{take } r = \lceil \log x \rceil \]

\[ \leq 4 \cdot 2^{\log x} + \frac{x}{\log \log x} \]

\[ = O \left( 2^{\log x} \right) + \frac{x}{\log \log x} \]

\[ \leq o \left( \frac{x}{\log \log x} \right) + \frac{x}{\log \log x} \quad \text{as } \log 2 < 1 \]

\[ = O \left( \frac{x}{\log \log x} \right) . \]
4. (20pts) Let \( \langle x \rangle = \lfloor x \rfloor + 1/2 \) denote the nearest integer to \( x \).

(i) If \( x \) is a real number then show that
\[
\lfloor x \rfloor = \langle x/2 \rangle + \lfloor x/2 \rfloor.
\]

Suppose that \( \{ x/2 \} < 1/2 \). Then
\[
\langle x/2 \rangle = \lfloor x/2 \rfloor \quad \text{and} \quad \lfloor x \rfloor = 2\lfloor x/2 \rfloor
\]
and so
\[
\lfloor x \rfloor = 2\lfloor x/2 \rfloor
\]
\[
= \lfloor x/2 \rfloor + \langle x/2 \rangle.
\]

Now suppose that \( \{ x/2 \} \geq 1/2 \). Then
\[
\langle x/2 \rangle = \lfloor x/2 \rfloor + 1 \quad \text{and} \quad \lfloor x \rfloor = 2\lfloor x/2 \rfloor + 1
\]
and so
\[
\lfloor x \rfloor = 2\lfloor x/2 \rfloor + 1
\]
\[
= \lfloor x/2 \rfloor + \langle x/2 \rangle.
\]

(ii) Let \( p_1, p_2, \ldots, p_m \) be the first \( m \) odd primes and let \( P(x, m) \) be the number of odd integers at most \( x \) and not divisible by any of these primes.

Show that
\[
P(x, m) = \sum_a \langle x/2a \rangle - \sum_b \langle x/2b \rangle
\]
where \( a \) and \( b \) run over all products of an even and an odd number of primes among \( p_1, p_2, \ldots, p_m \) respectively.

\[
P(x, m) = A(x, m + 1) + 1
\]
\[
= \lfloor x \rfloor - \lfloor x/2 \rfloor - \left( \sum \lfloor x/p_i \rfloor - \sum \lfloor x/2p_i \rfloor \right) + \left( \sum \lfloor x/p_ip_j \rfloor - \sum \lfloor x/2p_ip_j \rfloor \right) + \ldots
\]
\[
= \langle x/2 \rangle - \sum \langle x/p_i \rangle + \sum \langle x/p_ip_j \rangle + \ldots
\]
\[
= \sum_a \langle x/2a \rangle - \sum_b \langle x/2b \rangle.
\]
(iii) Show that
\[ \pi(x) = \pi(\sqrt{x}) + P(x, \pi(\sqrt{x}) - 1) - 1. \]

Let \( r = \pi(\sqrt{x}) \). Then
\[ \pi(x) = r + A(x, r) \]
\[ = \pi(\sqrt{x}) + P(x, \pi(\sqrt{x}) - 1). \]

(iv) Use (iii) to calculate \( \pi(200) \).

Now the odd primes up to 14 are 3, 5, 7, 11 and 13. Thus
\[ \pi(\sqrt{200}) = 6. \]

On the other hand, one can compute
\[ P(200, 5) = 100 - (33 + 20 + 14 + 9 + 8) + (7 + 5 + 3 + 3 + 2 + 2 + 1 + 1 + 1) - (1 + 1 + 1) = 41. \]

Thus
\[ \pi(200) = \pi(14) + P(200, 5) - 1 \]
\[ = 6 + 41 - 1 \]
\[ = 46. \]
5. (10pts) Derive the prime number theorem from the relation
\[ \vartheta(x) \sim x. \]

We have
\[
x \sim \sum_{p \leq x} \log p \\
\leq \sum_{p \leq x} \log x \\
= \log x \sum_{p \leq x} 1 \\
= \pi(x) \log x.
\]

On the other hand
\[
x \sim \sum_{x^{1-\epsilon} \leq p \leq x} \log p \\
\geq \sum_{x^{1-\epsilon} \leq p \leq x} \log x^{1-\epsilon} \\
= (1 - \epsilon) \log x \sum_{x^{1-\epsilon} \leq p \leq x} 1 \\
= (1 - \epsilon) (\pi(x) - \pi(x^{1-\epsilon})) \log x \\
= (1 - \epsilon) (\pi(x) + O(x^{1-\epsilon})) \log x.
\]

Putting these together, we see that
\[ \pi(x) \sim \frac{x}{\log x}. \]
**Bonus Challenge Problems**

6. (10pts) Show that

\[
\int_2^x \frac{\pi(t)}{t^2} \, dt = \sum_{p \leq x} \frac{1}{p} + o(1).
\]

We apply partial summation to

\[
\lambda_n = p_n \quad c_n = 1 \quad \text{and} \quad f(x) = \frac{1}{x}.
\]

We get

\[
\sum_{p \leq x} \frac{1}{p} = \frac{\pi(x)}{x} - \int_2^x \frac{\pi(t)}{t^2} \, dt.
\]

As

\[
\pi(x) = O\left(\frac{x}{\log \log x}\right)
\]

the first expression on the RHS is certainly \( o(1) \). Rearranging we get

\[
\int_2^x \frac{\pi(t)}{t^2} \, dt = \sum_{p \leq x} \frac{1}{p} + o(1).
\]
7. (10pts) Show that there are constants $c_1$ and $c_2$ such that

$$c_1 \frac{x}{\log x} < \pi(x) < c_2 \frac{x}{\log x}.$$ 

See lecture 7.