MODEL ANSWERS TO THE FIRST HOMEWORK

6.1.1. We have
\[ \sigma_k(n) = \sum_{d|n} d^k. \]
Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be the function \( f(n) = n^k \). If \( m \) and \( n \) are any two natural numbers then
\[
\begin{align*}
  f(mn) &= (mn)^k \\
  &= m^k n^k \\
  &= f(m)f(n).
\end{align*}
\]
In particular \( f \) is multiplicative. Therefore \( \sigma_k(n) \) is multiplicative.
Suppose that \( n = p^e \) is a power of a prime. Then the divisors of \( n \) are the powers of \( p \) up to \( n \) and so
\[
\begin{align*}
  \sigma_k(p^e) &= \sum_{i=0}^{e} (p^i)^k \\
  &= \sum_{i=0}^{e} p^{ik} \\
  &= \sum_{i=0}^{e} (p^k)^i \\
  &= \frac{p^{k(e+1)} - 1}{p^k - 1}.
\end{align*}
\]
Thus if \( n = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r} \) is the prime factorisation then
\[
\sigma_k(n) = \prod_{i=1}^{r} \frac{p_i^{k(e_i+1)} - 1}{p_i^k - 1}.
\]

6.1.2. We first show that both sides are multiplicative. As \( \tau \) is a multiplicative function it follows that \( \tau^3 \) is a multiplicative function and so the LHS is a multiplicative function.
Similarly
\[
\sum_{d|n} \tau(d)
\]
is multiplicative as \( \tau \) is multiplicative and so the RHS is multiplicative, as the square of a multiplicative function is multiplicative.
Suppose that $n = p^e$ is a power of a prime. For the LHS we have

$$
\sum_{d \mid p^e} \tau^3(d) = \sum_{i=0}^{e} \tau^3(p^i)
= \sum_{i=0}^{e} (1 + i)^3
= \sum_{i=1}^{e+1} i^3
= (e + 1)^2(e + 2)^2 \frac{4}{4}.
$$

On the other hand, for the RHS we have

$$
\left( \sum_{d \mid p^e} \tau(d) \right)^2 = \left( \sum_{i=0}^{e} \tau(p^i) \right)^2
= \left( \sum_{i=0}^{e} (1 + i) \right)^2
= \left( \sum_{i=1}^{e+1} i \right)^2
= \left( \frac{(e + 1)(e + 2)}{2} \right)^2.
$$

Thus we have equality when $n$ is a power of a prime.

Suppose that $p_1^{e_1}p_2^{e_2} \ldots p_k^{e_k}$ is the prime factorisation of $n$. Suppose the LHS is the function $l(n)$ and the RHS is the function $r(n)$. We have

$$
l(n) = l(p_1^{e_1}p_2^{e_2} \ldots p_k^{e_k})
= l(p_1^{e_1})l(p_2^{e_2}) \ldots l(p_k^{e_k})
= r(p_1^{e_1})r(p_2^{e_2}) \ldots r(p_k^{e_k})
= r(p_1^{e_1}p_2^{e_2} \ldots p_k^{e_k})
= r(n).
$$

6.1.3. Suppose that $n = p^e$ is a prime power. Then

$$
\sigma(n) = 1 + p + p^2 + \cdots + p^e.
$$

Hence $\sigma(n)$ is odd if and only if

$$
p + p^2 + \cdots + p^e
$$

is even. If $p = 2$ then $\sigma(n)$ is always odd and $p$ is odd then $e$ is even.
If \( n = 2^{e_0}p_1^{e_1}p_2^{e_2} \ldots p_r^{e_r} \) is the prime factorisation of \( n \) then \( \sigma(p_i^{e_i}) \) is odd by multiplicativity so that we must have \( e_i \) is even for all \( 0 < i \leq r \). If \( e_0 \) is even it follows that \( n \) is a square and if \( e_0 \) is odd then \( n \) is twice a square.

6.1.5. If we expand the RHS we get

\[
\left( \sum_{n=1}^{\infty} \frac{1}{n^s} \right)^2 = \left( \sum_{m=1}^{\infty} \frac{1}{m^s} \right) \left( \sum_{n=1}^{\infty} \frac{1}{n^s} \right)
\]

\[
= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^s n^s}
\]

\[
= \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \frac{1}{(md)^s}
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{d|n} \frac{1}{n^s} \right)
\]

\[
= \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}.
\]

Here we used the fact that since the series converges and all terms are positive, it follows that the series converges absolutely, so that we are free to rearrange the terms of the sum.

6.1.6. (a) Let \( d_1 \) be an odd divisor of \( n \). Suppose that \( n = 2^e m \) where \( m \) is odd. We have

\[
\sum_{i=0}^{e} (-1)^{n/2^i} d_1 2^i d_1 = (-1)^{n/d_1} d_1 (1 + 2 + 2^2 + \cdots + 2^{e-1} - 2^e)
\]

\[
= (-1)^{n/d_1} d_1 \left( \frac{2^e - 1}{2 - 1} - 2^e \right)
\]

\[
= -(-1)^{n/d_1} d_1.
\]

Thus

\[
- \sum_{d|n} (-1)^{n/d} = \sum_{d_1|m} - \sum_{i=0}^{e} (-1)^{n/2^i} d_1 2^i d_1
\]

\[
= \sum_{d_1|m} d_1.
\]

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(b) Suppose that $n$ is even, so that $n = 2l$ for some natural number $l$ and $e > 0$.

\[
2\sigma(l) - \sigma(n) = \sum_{d|l} 2d - \sum_{d|n} d \\
= \sum_{2d|l} d - \sum_{d|n} d \\
= \sum_{d_1|m} d_1 \\
= \sum_{d|n} (-1)^{n/d} d.
\]

6.1.8. Suppose that $m$ and $n$ are coprime. If $m$ is divisible by $\mu$ distinct primes and $n$ is divisible by $\nu$ distinct primes then $mn$ is divisible by $\mu + \nu$ distinct primes, so that

\[
\omega(mn) = \omega(m) + \omega(n)
\]

and so $\omega$ is additive.

6.1.9. If $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ is the prime factorisation of $n$ and $p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_r$ then clearly

\[
p_k \geq k + 1.
\]

Note also that $r = \omega(n)$. If $n = p^e$ is a power of a prime and $p \geq k$ then

\[
\varphi(n) = p^e - p^{e-1} \\
= n - \frac{n}{p} \\
= n(1 - \frac{1}{p}) \\
\geq n(1 - \frac{1}{k}).
\]
Thus

\[ \varphi(n) = \varphi(p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}) \]
\[ = \varphi(p_1^{e_1}) \varphi(p_2^{e_2}) \ldots \varphi(p_r^{e_r}) \]
\[ \geq \prod_{k=1}^{\omega(n)} p^e_i \left(1 - \frac{1}{k+1}\right) \]
\[ = n \prod_{k=2}^{\omega(n)+1} (1 - \frac{1}{k}) \]
\[ = n \prod_{k=2}^{\omega(n)+1} \frac{k-1}{k} \]
\[ = \frac{n}{\omega(n) + 1}. \]

To establish the inequalities

\[ 2^{\omega(n)} \leq \tau(n) \leq n, \]

note that all three terms are positive and multiplicative. So we may assume that \( n = p^e \) is a power of a prime. In this case \( \omega(n) = 1 \), unless \( e = 0 \) so that \( n = 1 \) and \( \tau(n) = 1 + e \) and the inequalities are clear.

Taking logs we see that

\[ 2^{\omega(n)} \leq n \quad \text{implies that} \quad \omega(n) \leq \frac{\log n}{\log 2}. \]

Note that \( \omega(n) + 1 \leq 2\omega(n) \) and so

\[ \varphi(n) \geq \frac{\log 2}{2\log n} n. \]

6.2.1. We have that

\[ \sigma(n) = \sum_{d \mid n} d \]

so that this result follows by Möbius inversion.

6.2.2. We may write \( n \) in the form \( n = n_1^2 n_2 \) where \( n_2 \) is square-free.

Suppose that \( d^2 \mid n \). Suppose that \( p \) is a prime and \( p^e \) divides \( d \). Then \( p^{2e} \) divides \( n \) and so \( p^{2e-1} \) divides \( n_1 \), since \( p^2 \) does not divide \( n_2 \). But then \( p^e \) divides \( n_1 \). Thus \( d \) divides \( n_1 \).

In this case \( d^2 \mid n \) if and only if \( d \mid n_1 \). It follows that

\[ \sum_{d^2 \mid n} \mu(d) = \sum_{d \mid n_1} \mu(d). \]
But

\[ \sum_{d|n_1} \mu(d) = \begin{cases} 1 & \text{if } n_1 = 1 \\ 0 & \text{otherwise.} \end{cases} \]

But \( \mu(n) \) is zero if and only if \( n \) is square-free. Thus

\[ |\mu(n)| = \begin{cases} 1 & \text{if } n_1 = 1 \\ 0 & \text{otherwise.} \end{cases} \]