6.1.10. If \( x \in S \) then let \( \chi_x : S \to \mathbb{C} \) be the function

\[
\chi_x(y) = \begin{cases} 
1 & \text{if } y = x \\
0 & \text{otherwise}.
\end{cases}
\]

Note that

\[
f = \sum_{x \in X} f(x) \chi_x.
\]

Indeed both sides are functions \( S \to \mathbb{C} \) and it is easy to see they have the same effect on every element of \( S \).

Thus we are reduced to the case \( f = \chi_x \). In this case, the fact that both sides are equal is shown in the proof of inclusion-exclusion.

Now suppose that \( S \) consists of \( N \) real numbers. Note that if \( i < j \) then

\[
S_i \cap S_j = S_i
\]

so that \( \bigcap_{j=1}^{k} S_{i_j} = S_m \),

where \( m \) is the smallest index. Note also that

\[
\sum_{s \in S_i} f(s) = \sum_{i=1}^{i} f(x_i)
\]

\[
= f(x_1) + f(x_2) + \cdots + f(x_i)
\]

\[
= x_1 + (x_2 - x_1) + \cdots + (x_i - x_{i-1})
\]

\[
= x_i,
\]

the largest element of \( S_i \).

The union of the sets \( S_i \) is the whole of \( S \), so the LHS is zero. The first sum on the RHS is \( X_N \). If one brings this over to the LHS then this gives a proof of the formula (2.4.4.a).

6.2.3. We check that both sides of this equation are additive. If \( x \) and \( y \) are any positive reals then

\[
\log xy = \log x + \log y,
\]

so that the LHS is certainly additive. For the RHS, note that the only non-zero terms of the sum

\[
\sum_{d \mid n} \Lambda(d)
\]

are when \( d \) is a power of a prime. If \( m \) and \( n \) are coprime and \( d \) divides \( mn \) then we can write \( d = d_1d_2 \) where \( d_1 \) divides \( m \) and \( d_2 \) divides \( n \).
If \( d \) is a pure power of a prime then one of \( d_1 \) and \( d_2 \) is one and the other is equal to \( d \). It follows that the RHS is additive as well. Thus we may assume that \( n = p^e \) is a power of a prime \( p \). In this case

\[
\sum_{d|n} \Lambda(d) = \sum_{i=0}^{e} \Lambda(p^i) \\
= \sum_{i=1}^{e} \log p \\
= e \log p \\
= \log p^e \\
= \log n.
\]

By Möbius inversion, we have

\[
\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d) \\
= \sum_{d|n} \mu(d) (\log n - \log d) \\
= \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log d \\
= \log n M(n) - \sum_{d|n} \mu(d) \log d \\
= - \sum_{d|n} \mu(d) \log d
\]

Thus

\[
\sum_{d|n} \mu(d) \log d = -\Lambda(n).
\]

6.2.4. Note that \( \mu(n) \) is multiplicative and the product of multiplicative functions is multiplicative, so that the LHS is multiplicative. On the other hand, if \( m \) and \( n \) are coprime then the prime factors of \( mn \) are simply the prime factors of \( m \) and \( n \). It is easy to see that the RHS is also multiplicative.
Thus we are reduced to the case when \( n = p^e \) is the power of a prime \( p \). In this case

\[
\sum_{d|n} \mu(d) f(d) = \sum_{i=0}^{e} \mu(p^i) f(p^i)
\]

\[
= \mu(1) f(1) + \mu(p) f(p) + 0 + \cdots + 0
\]

\[
= 1 - f(p).
\]

6.2.7. We have

\[
\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{d=1}^{\infty} \frac{\mu(d)}{(md)^s} = \sum_{d|n} \frac{\mu(d)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \mu(d) = \sum_{n=1}^{\infty} \frac{1}{n^s} M(n) = 1,
\]

where we used the fact that both

\[
\sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}
\]

are absolutely convergent, so that we can rearrange the terms of the sums.

6.2.8. (a) Let \( I \) be the set of natural numbers from one to \( n \) and let \( I^k \) be the Cartesian product of \( I \) with itself \( k \) times, so that \( I^k \) has cardinality \( n^k \). Let

\[ C_d = \{ (a_1, a_2, \ldots, a_k) \in I^k \mid \text{the greatest common divisor of } a_1, a_2, \ldots, a_k \text{ and } n \text{ is } d \}. \]

Note that \( C_d \) is a partition of \( I^k \) indexed by the divisors of \( n \). If \( (a_1, a_2, \ldots, a_k) \in C_d \) then \( a_i \) is divisible by \( d \) so that \( a_i = b_id \). In this
case \((b_1, b_2, \ldots, b_k) \in J_k(n/d)\), so that

\[ |C_{n/d}'| = J_k(d). \]

Thus

\[ n^k = |I^k| \]
\[ = \sum_{d|n} |C_{n/d}| \]
\[ = \sum_{d|n} J_k(d). \]

(b) As \(F(n) = n^k\) is multiplicative, it follows that \(J_k(n)\) is multiplicative.

(c) We already showed that the LHS is multiplicative in (b). The RHS is the product of \(n^k\), which is multiplicative and the other term is also multiplicative. Thus we are reduced to the case when \(n = p^e\) is a power of a prime.

In this case the condition that the greatest common divisor is coprime to \(n\) is the same as the condition that the greatest common divisor is not divisible by \(p\). The number of \(k\)-tuples whose greatest common divisor is divisible by \(p\) is

\[ (n/p)^k. \]

Thus the number of \(k\)-tuples whose greatest common divisor is not divisible by \(p\) is

\[ J_k(p^e) = n^k - (n/p)^k \]
\[ = n^k \left( 1 - \frac{1}{p^k} \right) \]
\[ = n^k \prod_{p|n} \left( 1 - \frac{1}{p^k} \right). \]
6.2.9. We have

$$\sum_{d_1, d_2 = m, e_1, e_2 = n} \mu(d_1)\mu(e_1) F(d_2, e_2) = \sum_{d_1, d_2 = m, e_1, e_2 = n} \mu(d_1)\mu(e_1) \sum_{d, e | d_2, e_2} f(d, e)$$

$$= \sum_{d, e | d, e | n} f(d, e) \sum_{d_1, d_2 | m, e_1, e_2} \mu(d_1)\mu(e_1)$$

$$= \sum_{d, e | d, e | n} f(d, e) \sum_{d_1, d_2 | m, e_1, e_2} \mu(d_1) \sum_{e_1, e_2 | n} \mu(e_1)$$

$$= \sum_{d, e | d, e | n} f(d, e) M(m/d) M(n/e)$$

$$= f(m, n).$$

6.3.2. Note that

$$\downarrow 2x - 2\downarrow x$$

is either 0 or 1. On the other hand,

$$\downarrow x + \downarrow y + 1 = \downarrow x + \downarrow y + 1$$

$$\geq \downarrow x + y.$$ 

Suppose that

$$\downarrow 2x - 2\downarrow x = 1.$$

We have

$$\downarrow 2x + \downarrow 2y \geq \downarrow 2x + \downarrow 2y + 1$$

$$= \downarrow x + \downarrow y + \downarrow x + \downarrow y + 1$$

$$\geq \downarrow x + \downarrow y + \downarrow x + \downarrow y.$$ 

By symmetry we are also done if

$$\downarrow 2y - 2\downarrow y = 1.$$

Thus we may assume that

$$\downarrow 2x - 2\downarrow x = 0 \quad \text{and} \quad \downarrow 2y - 2\downarrow y = 0.$$

Note that if we multiply

$$x = \downarrow x + \{ x \}$$

by 2 we get

$$2x = 2\downarrow x + 2\{ x \}$$

so that

$$\downarrow 2x = 2\downarrow x + 2\{ x \},$$

and so $2\{ x \} < 1$, that is, $\{ x \} < 1/2.$
In this case, since

\[ x + y = \lfloor x \rfloor + \lfloor y \rfloor + \{ x \} + \{ y \} \]

and \( \{ x \} + \{ y \} < 1 \), we see that

\[ \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor. \]

Thus

\[ \lfloor 2x \rfloor + \lfloor 2y \rfloor = 2\lfloor x \rfloor + 2\lfloor y \rfloor \]
\[ = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x \rfloor + \lfloor y \rfloor \]
\[ = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor. \]

6.3.4. (a) One direction is clear. If

\[ x = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}, \]

then \( x \) is rational, since the rationals are closed under addition. Now suppose that \( x \) is rational. Then we may write

\[ x = \frac{p}{q} \]

where \( p \) and \( q > 0 \) are coprime integers. We would like to write down an invariant that goes down at each step.

It is tempting to try to use the denominator. If we start with \( x = 2/3 \) then \( a_1 = 1/2 \) and \( x_2 = 1/6 \) and in this case the denominator increases. It is tempting to believe that perhaps the denominator always divides \( q! \). But if \( x = 3/7 \) then we get

\[ \frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}, \]

and 231 does not divide 7!. There does not seem to be a simple invariant only depending on the denominator that goes down.

In fact it is easy to see that the numerator always goes down. In fact if we write \( q = cp + d \), where \( 0 \leq d \leq p - 1 \) then \( c \geq 1 \) as \( x \leq 1 \) implies \( q \geq p \). We claim that \( a_1 = c + 1 \). Indeed

\[ x - \frac{1}{c} = \frac{p}{q} - \frac{1}{c} \]
\[ = \frac{pc - q}{qc} \]
\[ = \frac{-d}{qc} \]
\[ < 0, \]

6
so that \( a_1 \geq c + 1 \). But

\[
\begin{align*}
x - \frac{1}{c + 1} &= \frac{p}{q} - \frac{1}{c + 1} \\
&= \frac{p(c + 1) - q}{q(c + 1)} \\
&= \frac{p - d}{q(c + 1)} \\
&\geq 0.
\end{align*}
\]

Thus \( a_1 = c + 1 \) and

\[
x_1 = \frac{p - q}{q(c + 1)}.
\]

It follows that the numerator of \( x_1 \) is at most \( p - q < p \).
Thus the algorithm terminates in at most \( p \) steps if \( x \) has numerator \( p \).

(b) Note that the sum of the reciprocals of \( b_1, b_2, \ldots \) converges by assumption. Let

\[
x = \sum_{n=1}^{\infty} \frac{1}{b_n}.
\]

We show by induction that if we choose the natural numbers \( a_1, a_2, \ldots \) as in part (a) then in fact \( a_n = b_n \) for all \( n \). Suppose that we have proved this result up to \( n \). Clearly

\[
x - \sum_{k\leq n} \frac{1}{a_k} = x - \sum_{k\leq n} \frac{1}{b_k} \\
= \sum_{k>n} \frac{1}{b_k} \\
> \frac{1}{b_{n+1}},
\]

7
so that $a_{n+1} < b_{n+1}$. On the other hand,
\[
x - \sum_{k \leq n} \frac{1}{a_k} = x - \sum_{k \leq n} \frac{1}{b_k} = \sum_{k > n} \frac{1}{b_k} = \frac{1}{b_{n+1}} + \sum_{k > n+1} \frac{1}{b_k} < \frac{1}{b_{n+1}} + \frac{1}{b_{n+1} - 1} - \frac{1}{b_{n+1}} = \frac{1}{b_{n+1} - 1}.
\]
Thus $a_{n+1} \geq b_{n+1}$. It follows that $a_{n+1} = b_{n+1}$ and this completes the induction.

As $a_1, a_2, \ldots = b_1, b_2, \ldots$ is an infinite sequence, it follows that the algorithm never terminates and so $x$ is irrational.

Let $b_k = 2^{3^k} + 1$. We have
\[
\sum_{n \geq k} \frac{1}{b_n} = \sum_{n \geq k} \frac{1}{2^{3^n} + 1} \leq \sum_{n \geq k} \frac{1}{2^{3^n}} = \frac{1}{2^{3^k}} \leq \frac{1}{23^{k-1}} = \frac{1}{b_{k-1} - 1}.
\]
Thus
\[
\sum (2^{3^k} + 1)^{-1}
\]
is irrational.

6.3.6. (a) We prove that
\[
\frac{(ab)!}{a!b!^a}
\]
is a natural number, for all natural numbers $a$ and $b$. We proceed by induction on $a$. 8
If \( a = 1 \) then
\[
\frac{(ab)!}{a!(b!)^a} = \frac{b!}{1!(b!)^1} = 1.
\]

Now suppose the result holds for \( a \).
\[
\frac{((a+1)b)!}{(a+1)!(b!)^{a+1}} = \frac{(ab)! (a+1)b((a+1)b-1)!}{a!(b!)^a (ab)!(a+1)b!} = \frac{(ab)! ((a+1)b-1)!}{a!(b!)^a (ab)!(b-1)!} = \frac{(ab)! (((a+1)b-1)!)}{a!(b!)^a (b-1)!}.
\]

The first term is an integer by induction and the second term is an integer, since it is a binomial.

(b) Pick a prime \( p \). The exponent of the largest power of \( p \) dividing \((2a)!\) is
\[
\frac{2a}{p} + \frac{2a}{p^2} + \ldots.
\]

Thus the exponent of the largest power of \( p \) dividing \((2a)!(2b)!\) is
\[
\frac{2a}{p} + \frac{2b}{p} + \frac{2a}{p^2} + \frac{2b}{p^2} + \ldots.
\]

On the other hand the exponent of the largest power of \( p \) dividing \(a!b!(a+b)!\) is
\[
\frac{a}{p} + \frac{b}{p} + \frac{a+b}{p} + \frac{a}{p^2} + \frac{b}{p^2} + \frac{a+b}{p^2} + \ldots.
\]

The difference of the exponent of the numerator and the denominator is then a sum of terms of the form
\[
\frac{2a}{p^k} - \frac{2b}{p^k} - \frac{a}{p^k} - \frac{b}{p^k} - \frac{a+b}{p^k}.
\]

By (6.3.2) applied to \( \alpha = \frac{a}{p^k} \) and \( \beta = \frac{b}{p^k} \) this is non-negative. But then
\[
\frac{(2a)!(2b)!}{a!b!(a+b)!}
\]

is a natural number, for all natural numbers \( a \) and \( b \) (since it’s denominator is not divisible by any prime).
6.4.2. Note that
\[
\sum_{i,j=1:p_i<p_j}^{\infty} \frac{x}{p_ip_j} = \sum_{p_ip_j \leq x, p_i<p_j} \frac{x}{p_ip_j} \\
\leq \sum_{p_ip_j \leq x, p_i<p_j} \frac{x}{p_ip_j} = E,
\]
where
\[
E \leq \sum_{p_ip_j \leq x, p_i<p_j} 1.
\]
The term on the RHS is the number of ways to pick two primes \(p_i, p_j\) such that \(p_i < p_j\) and \(p_ip_j \leq x\). Let \(y = p_ip_j\). Then \(y\) determines \(p_i\) and \(p_j\) by unique factorisation and \(y\) is a natural number between 1 and \(x\) so that \(y \leq \lfloor x \rfloor \leq x\).
Thus \(E\) is at most \(x\) and so \(-E = O(1)\).