6.8.1. We know that
\[
\lim_{s \to 1^+} \prod_q \left(1 - \frac{1}{q^{-s}}\right)^{-1} = \infty.
\]
If we take logs we get
\[
\lim_{s \to 1^+} \sum_q \log \left(1 - \frac{1}{q^{-s}}\right) = -\infty.
\]
As
\[
\log \left(1 - \frac{1}{q^{-s}}\right) \geq -\frac{2}{q^{-s}},
\]
this implies
\[
\lim_{s \to 1^+} \sum_q -\frac{2}{q^{-s}} = -\infty.
\]
Thus
\[
\lim_{s \to 1^+} \sum_q \frac{1}{q^{-s}} = \infty.
\]
This implies
\[
\sum_q \frac{1}{q},
\]
diverges.
6.8.2. In the course of the proof of (9.2) we established that
\[
\lim_{s \to 1^+} (s - 1)\zeta(s) = 1.
\]
We also proved that \(L(s)\) is continuous at \(s = 1\) and that \(L(1) \neq 0\).
Thus
\[
\lim_{s \to 1^+} (s - 1)\zeta(s)L(s) = \lim_{s \to 1^+} (s - 1)\zeta(s) \cdot \lim_{s \to 1^+} L(s)
\]
\[
= L(1)
\]
\[
\neq 0.
\]
As we showed
\[
\zeta(s)L(s) = \left(1 - \frac{1}{2^s}\right)^{-1} \prod_q \left(1 - \frac{1}{q^s}\right)^{-2} \cdot \prod_r \left(1 - \frac{1}{r^{2s}}\right)^{-1},
\]
\[
1
\]
it follows that

\[ L(1) = \lim_{s \to 1^+} \left( 1 - \frac{1}{2^s} \right)^{-1} \cdot \lim_{s \to 1^+} \prod_{q} \left( 1 - \frac{1}{q^s} \right)^{-2} \cdot \lim_{s \to 1^+} \prod_{r} \left( 1 - \frac{1}{r^{2s}} \right)^{-1}. \]

We proved that the last functions is continuous and non-zero at 1. As the first function is also continuous and non-zero at 1, it follows that

\[ \lim_{s \to 1^+} (s - 1) \prod_{q} \left( 1 - \frac{1}{q^s} \right) = A \]

where \( A \) is non-zero.

Thus

\[ \lim_{s \to 1^+} (s - 1) \prod_{q} \left( 1 - \frac{1}{q^s} \right)^{-1} = 0. \]

Suppose that the limit

\[ \lim_{s \to 1^+} \prod_{r} \left( 1 - \frac{1}{r^s} \right)^{-1} = B \]

exists and is finite. As

\[ \zeta(s) = \left( 1 - \frac{1}{2^s} \right)^{-1} \prod_{q} \left( 1 - \frac{1}{q^s} \right)^{-1} \cdot \prod_{r} \left( 1 - \frac{1}{r^s} \right)^{-1}, \]

Then

\[ 1 = \lim_{s \to 1^+} \left( 1 - \frac{1}{2^s} \right)^{-1} \lim_{s \to 1^+} (s - 1) \prod_{q} \left( 1 - \frac{1}{q^s} \right)^{-1} \cdot \lim_{s \to 1^+} \prod_{r} \left( 1 - \frac{1}{r^s} \right)^{-1} \]

\[ = 2 \cdot 0 \cdot B \]

\[ = 0, \]

a contradiction.

Thus

\[ \lim_{s \to 1^+} \prod_{r} \left( 1 - \frac{1}{r^s} \right)^{-1} \]

diverges. Arguing as in (6.8.1) it follows that the sum of the reciprocals of the primes congruent to 3 modulo 4 diverges.
6.8.3. We have
\[
\prod_p \sum_{k=0}^{\infty} f(p^k) = \sum_{r,p_1,p_2,\ldots,p_r,k_1,k_2,\ldots,k_r} f(\prod_{i=1}^{r} p_i^{k_i})
\]
\[
= \sum_{r,p_1,p_2,\ldots,p_r,k_1,k_2,\ldots,k_r} f(\prod_{i=1}^{r} p_i^{k_i})
\]
\[
= \sum_{n=1}^{\infty} f(n),
\]
where we are allowed to rearrange the sum by absolute convergence.
If \( f(n) \) is completely multiplicative and \( f(p) = 1 \) then \( f(p^k) = 1 \) for every \( k \) so that
\[
\sum_{k=0}^{\infty} f(p^k)
\]
diverges, a contradiction. We have
\[
\prod_p (1 - f(p))^{-1} = \prod_p \sum_{k=0}^{\infty} f(p)^k
\]
\[
= \prod_p \sum_{k=0}^{\infty} f(p^k),
\]
and so we done by the first part.
2. (a) Suppose that
\[
g(x) = \frac{p(x)}{q(x)} e^{-1/x^2},
\]
for \( x \neq 0 \), where \( p(x) \) and \( q(x) \) are polynomials. Then
\[
g'(x) = \frac{p'(x)q(x)e^{-1/x^2} + 2p(x)q(x)e^{-1/x^2}/x^3 - q'(x)p(x)e^{-1/x^2}}{q^2(x)}
\]
\[
= \frac{p'(x)q(x)x^3 + 2p(x)q(x) - q'(x)p(x)x^3}{q^2(x)x^3} e^{-1/x^2}.
\]
Thus we are done by induction on \( n \).
(b) Suppose that
\[
g(x) = \frac{p(x)}{q(x)} e^{-1/x^2},
\]
for \( x \neq 0 \), where \( p(x) \) and \( q(x) \) are polynomials and \( g(0) = 0 \). We first check that
\[
\lim_{x \to 0} \frac{p(x)}{q(x)} e^{-1/x^2} = 0.
\]
Factoring $p(x)$ and $q(x)$ it suffices to prove that
\[
\lim_{x \to 0} x^k e^{-1/x^2} = 0,
\]
for any integer $k$. Replacing $x$ by $1/x$ it suffices to prove that
\[
\lim_{x \to \infty} x^k e^{-x^2} = 0,
\]
for any integer $k$. This is easy (and well-known).

We now compute the derivative of $g(x)$.
\[
\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{p(x)}{q(x)} e^{-1/x^2}
= 0.
\]

Thus
\[
f^{(n)}(0) = 0.
\]

by induction on $n$.

(c) The Taylor series for $f(x)$ has all of its coefficients zero. This surely converges everywhere and defines the zero function but this is not equal to $f(x)$. 