PRACTICE PROBLEMS FOR THE FIRST MIDTERM

1. Give the definition of:
   (i) \( \tau \).
   (ii) \( \sigma \).
   (iii) \( \pi(x) \).
   (iv) a multiplicative function.
   (v) a perfect number.
   (vi) a Mersenne prime.
   (vii) The Möbius function.
   (viii) the round down.
   (ix) the fractional part.
   (x) \( f(x) = O(g(x)) \).
   (xi) \( f(x) = o(g(x)) \).
   (xii) \( f(x) \sim g(x) \).
   (xiii) Euler’s constant.

2. Show that if \( f: \mathbb{N} \rightarrow \mathbb{C} \) is a multiplicative function then
   \[
   F(n) = \sum_{d|n} f(d)
   \]
   is a multiplicative function.

3. Show that if \( n \) is an even perfect number then \( n = 2^{p-1}(2^p - 1) \), where \( 2^p - 1 \) is a Mersenne prime.

4. Show that the number of ordered pairs pairs of natural numbers whose lowest common multiple is \( n \) is \( \tau(n^2) \).

5. Show that if \( d|n \) and \( (n,r) = 1 \) then the number of solutions, modulo \( n \), of
   \( x \equiv r \mod d \) where \( (x,n) = 1 \),
   is
   \[
   \frac{\varphi(n)}{\varphi(d)} = \sum_{d|n} \mu(d)F(n/d).
   \]

6. Show that if \( f \) and \( F \) are as in (2) then (a)
   \[
   f(n) = \sum_{d|n} \mu(d)F(n/d).
   \]
   (b) if \( F \) is multiplicative then \( f \) is multiplicative.

7. Let \( f(x,n) \) be the number of integers less than or equal to \( x \) and coprime to \( n \). Prove that
(a) \[ \sum_{d|n} f\left(\frac{x}{d}, \frac{n}{d}\right) = \lfloor x \rfloor \]
(b) \[ f(x, n) = \sum_{d|n} \mu(d) \lfloor \frac{x}{d} \rfloor. \]

8. Show that
\[ \prod_{\substack{p \leq x}} \left(1 - \frac{1}{p}\right) < \frac{1}{\log x}. \]