10. The diffusion equation

We now turn our attention to the diffusion equation, so that we move from the study of hyperbolic equations to the study of parabolic equations. Recall that the one dimensional diffusion equation reads

$$u_t = ku_{xx}.$$

Of course the study of diffusion is very different from the study of waves.

It will turn out that writing down an explicit form of the solution to the diffusion equation is harder than for waves. In this section we focus on a useful property of solutions to the diffusion equation.

Theorem 10.1 (Maximum principle)**.** *Suppose that $u(x,t)$ is a solution to the diffusion equation in the rectangle $0 \leq x \leq l$, $0 \leq t \leq T$.*

If the maximum value of $u(x,t)$ is achieved at a point which is neither an initial point, $t = 0$, $0 \leq x \leq l$ nor a lateral point, $x = 0$ or $x = l$, $0 \leq t \leq T$, then u is constant.

This is the strong form of the maximum principle. Note that as u is continuous, it has a maximum somewhere in the rectangle. The strong form says that if u is not constant this maximum is never strictly inside the rectangle nor on the upper edge $t = T$ and $0 < x < wl$. The weak form simply says that the maximum is achieved on the sides or at the start, it does not preclude the possibility that the maximum is also achieved inside the rectangle or at the end.

There is also a companion minimum principle. The strong form says that the minimum is never achieved strictly inside the rectangle nor at the end, unless u is constant. The reader can formulate for themselves the weak form of the minimum principle.

In fact the minimum principle follows easily from the maximum principle applied to $v = -u$. As the diffusion equation is linear, v is also a solution to the diffusion equation. A minimum for u is a maximum for v and u is constant if and only if v is constant.

Finally note that the maximum principle make physical sense. If you have a metal rod then the maximum and minimum temperature occur either at the start or at the two endpoints (by assumption there is no heat source). If you have a cylinder of liquid and a red dye then either there is the most dye at the start or at one of the two ends of the cylinder.

Proof of (10.1)*.* We only prove the weak form of the maximum principle.

The idea is to use the standard fact from calculus that on the interior of the rectangle a maximum is at a critical point, a point where the partial derivatives are all zero, and at maximum we also have $u_{xx} \leq 0$.

If we knew that $u_{xx} < 0$, that is, if we knew that $u_{xx} \neq 0$, then we would have

$$\begin{aligned} 0 &= u_t \\ &= ku_{xx} \\ &< 0, \end{aligned}$$

which is surely not possible. We introduce a trick to deal with the possibility that $u_{xx} = 0$ at the maximum.

Let M be the maximum of $u(x,t)$ on the three sides, the initial side and the two lateral sides. Pick a positive constant $\epsilon > 0$ and let

$$v(x,t) = u(x,t) + \epsilon x^2.$$

Claim 10.2. *$v(x,t) \leq M + \epsilon l^2$ on the rectangle.*

Let's assume (10.2). We would have

$$\begin{aligned} u(x,t) &= v(x,t) - \epsilon x^2 \\ &\leq M + \epsilon(l^2 - x^2). \end{aligned}$$

Letting ϵ go to zero we get that $u(x,t) \leq M$.

We now turn to the proof of the claim:

Proof of (10.2). Note that

$$v(x,t) \leq M + \epsilon l^2$$

on the three sides $t = 0$, $x = 0$ and $x = l$.

Note that

$$\begin{aligned} v_t - kv_{xx} &= u_t - ku_{xx} - 2k\epsilon \\ &= -2k\epsilon \\ &< 0. \end{aligned}$$

If (x_0, t_0) is a maximum of $v(x,t)$ in the interior of R so that

$$0 < t_0 < T \qquad \text{and} \qquad 0 < x_0 < l$$

then we would have

$$v_t = 0 \qquad \text{and} \qquad v_{xx} < 0,$$

at (x_0, t_0). In this case

$$\begin{aligned} v_t - kv_{xx} &= 0 - kv_{xx} \\ &> 0, \end{aligned}$$

which is not possible.

Finally suppose that there were a maximum at the end (x_0, t_0), along the line $t = T$ and $0 < x_0 < l$. In this case $t_0 = T$, $v_x(x_0, T) = 0$ and $v_{xx}(x_0, T) \leq 0$.

On the other hand, as $v(x_0, T)$ is bigger than $v(x_0, T - \delta)$ we have

$$v_t(x_0, t_0) = \lim \frac{v(x_0, T) - v(x_0, T - \delta)}{\delta} \geq 0,$$

as δ approaches zero from above. We again get a contradiction to the inequality we established above.

But v must have a maximum somewhere on the rectangle R. By a process of elimination it must be on one of the two lateral sides or at the start. But then $v(x, t) \leq M + \epsilon l^2$. □

□

One can use the maximum principle to prove that solutions to the diffusion equation with Dirichlet boundary conditions are unique:

There is at most one solution to

$$\begin{aligned} u_t - k u_{xx} &= f(x, t) \qquad \text{for} \qquad 0 < x < l \quad \text{and} \quad t > 0 \\ u(x, 0) &= \phi(x) \\ u(0, t) &= g(t) \\ u(l, t) &= h(t), \end{aligned}$$

where f, ϕ, g and h are four given functions.

Suppose that u_1 and u_2 are two solutions to the diffusion equation with auxiliary conditions above. Let $w = u_1 - u_2$ be their difference. By linearity we have that

$$\begin{aligned} w_t - k w_{xx} &= 0 \qquad \text{for} \qquad 0 < x < l \quad \text{and} \quad t > 0 \\ w(x, 0) &= 0 \\ w(0, t) &= 0 \\ w(l, t) &= 0. \end{aligned}$$

By the maximum principle, the function $w(x, t)$ has it maximum on one of three sides, the start or the lateral sides. As w is zero on the sides, this maximum is zero. But then

$$w(x, t) \leq 0.$$

By the minimum principle, the function $w(x, t)$ has it minimum on the three sides, the start or the lateral sides. As w is zero on the sides, this minimum is zero. But then

$$w(x, t) \geq 0.$$

As w is bounded from below and above by zero, it follows that $w = 0$. But then $u_1 = u_2$.

Aliter: We give a completely different way to show uniqueness, using the *energy method.*

As before we consider the difference $w = u_1 - u_2$. As before w satisfies the diffusion equation with all auxiliary conditions zero. We multiply the diffusion equation by w

$$\begin{aligned} 0 &= 0 \cdot w \\ &= (w_t - kw_{xx})w \\ &= w_t w - kw_{xx}w \\ &= \frac{1}{2}(w^2)_t - (kw_x w)_x + kw_x^2. \end{aligned}$$

If we integrate over the interval $0 < x < l$ then we get

$$\int_0^l \frac{1}{2}(w^2)_t \, \mathrm{d}x - \Big[kw_x w \Big]_0^l + \int_0^l kw_x^2 \, \mathrm{d}x.$$

The middle term is zero, because of the boundary conditions and so we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^l \frac{1}{2} w^2 \, \mathrm{d}x = - \int_0^l kw_x^2 \, \mathrm{d}x \leq 0.$$

Thus the integral is decreasing and so

$$\int_0^l w^2(x,t) \, \mathrm{d}x \leq \int_0^l w^2(x,0) \, \mathrm{d}x.$$

The last integral is zero, as $w(x,0) = 0$. As the integral of a square is non-negative, it follows that

$$\int_0^l w^2(x,t) \, \mathrm{d}x = 0.$$

The only way that this is possible is if

$$w = 0,$$

so that $u_1 = u_2$.

We can also use these methods to derive stability results.

We first use the energy method. Suppose that $h = g = f = 0$ and suppose

$$u_1(x,0) = \phi_1(x) \qquad \text{and} \qquad u_2(x,0) = \phi_2(x).$$

It follows that $w = u_1 - u_2$ is the solution with initial data $\phi_1(x) - \phi_2(x)$. As above, this implies

$$\int_0^l (u_1(x,t) - u_2(x,t))^2 \, \mathrm{d}x \leq \int_0^l (\phi_1(x,t) - \phi_2(x,t))^2 \, \mathrm{d}x.$$

It follows that if ϕ_1 is close to ϕ_2 in the sense that the integral on the RHS is small then u_1 is close to u_2 in the sense that the integral on the LHS is small.

If we try to use the maximum principle we get a different notion of being close. If we try to apply the maximum principle the first thing to note is that $w = u_1 - u_2 = 0$ on the two lateral sides and $w = \phi_1 - \phi_2$ at the start. Thus the maximum on the three sides is achieved at the start and so by the maximum principle

$$u_1(x,t) - u_2(x,t) \leq \max |\phi_1 - \phi_2|.$$

The minimum principle says that

$$u_1(x,t) - u_2(x,t) \geq - \max |\phi_1 - \phi_2|.$$

Putting these both together

$$\max_{0 \leq x \leq l} |u_1(x,t) - u_2(x,t)| \leq \max_{0 \leq x \leq l} |\phi_1(x) - \phi_2(x)|.$$