11. The diffusion equation on the line

Our goal is to solve as explicitly as possible the diffusion equation on the whole line

\[ u_t = ku_{xx} \quad \text{for} \quad -\infty < x < \infty, \quad 0 < t < \infty, \]

with the initial condition

\[ u(x,0) = \phi(x). \]

The method of solution is quite different to previous methods. We start with a particular choice of \( \phi(x) \), solve for this and induce all of the other solutions for different choices of \( \phi(x) \) from this one solution.

We start with five different ways to go from one solution to diffusion equation to more solutions (ignoring the initial condition).

(a) The translate \( u(x - y, t) \) of a solution \( u(x, t) \) is another solution.

(b) Any derivative of a solution is a solution (derivative with respect to either \( x \) or \( t \)).

(c) A linear combination of solutions is a solution (by linearity).

(d) An integral of solutions is a solution. If \( S(x,t) \) is a solution of the diffusion equation then so is \( S(x - y, t) \) and so is

\[ v(x,t) = \int_{-\infty}^{\infty} S(x-y)g(y) \, dy \]

for any function \( g(y) \) (provided the integral converges).

(e) If \( u(x,t) \) is any solution to the diffusion equation then so is the dilated function

\[ v(x,t) = u(\sqrt{a}x, at) \]

for any \( a > 0 \).

Note that (d) is really a limiting form of (c). To check (e) we simply apply the chain rule

\[ v_t = \frac{\partial(at)}{\partial t} u_t = au_t. \]

Similarly

\[ v_x = \frac{\partial(\sqrt{a}t)}{\partial x} u_x = \sqrt{a}u_x, \]

so that

\[ v_{xx} = \sqrt{a}\sqrt{a}u_{xx} = au_{xx}. \]
We now write down a particular choice of $\phi(x)$.

**Definition 11.1** (Heaviside step function). The **Heaviside step function** is defined as

$$H(x) = \begin{cases} 
1 & \text{for } x > 0 \\
0 & \text{for } x < 0.
\end{cases}$$

Note that the Heaviside step function does not change under dilation.

We are looking for a solution $Q(x, t)$ to the diffusion equation with the initial data of the Heaviside step function,

$$Q(x, 0) = 1 \quad \text{for } x > 0 \quad Q(x, 0) = 0 \quad \text{for } x < 0.$$ 

There are three steps.

**Step 1** We assume that $Q(x, t)$ has a very special form

$$Q(x, t) = g(p) \quad \text{where} \quad p = \frac{x}{\sqrt{4kt}}.$$  

Why would we jump to this assumption? Well, there are two observations. A dilation of $Q(x, t)$ is still a solution to the wave equation and $H(x, t)$ is unchanged under dilations. It follows that any dilation of $Q(x, t)$ is a solution to the same PDE with the same auxiliary conditions. By uniqueness it follows that $Q(x, t)$ is unchanged under dilations.

Note that $x/\sqrt{t}$ is unchanged under dilations and in fact $Q(x, t)$ must be a function of $x/\sqrt{t}$. It is convenient to rescale and write $Q$ as a function of $p$.

**Step 2** We now figure out the ODE which $g$ has to satisfy. If we apply the chain rule then we get

$$Q_t = \frac{dg}{dp} \frac{\partial p}{\partial t} = -\frac{1}{2t} \frac{x}{\sqrt{4kt}} g'(p)$$

$$Q_x = \frac{dg}{dp} \frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}} g'(p)$$

$$Q_{xx} = \frac{dQ_x}{dp} \frac{\partial p}{\partial x} = \frac{1}{4kt} g''(p)$$

$$0 = Q_t - kQ_{xx} = -\frac{1}{2t} \left(pg'(p) + \frac{1}{2} g''(p)\right).$$  

Putting all of this together we get

$$g'' + 2pg' = 0.$$  

This gives us an integrating factor of

$$\exp(\int 2p \, dp) = \exp(p^2).$$  

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This gives
\[ g'(p) = c_1 \exp(-p^2) \]
so that
\[ Q(x, t) = g(p) = c_1 \int_0^{\sqrt{4\pi t}} e^{-p^2} \, dp + c_2. \]

**Step 3** Now we turn this expression into an explicit formula. If \( t \) tends to zero from above then the upper limit increases in magnitude. If \( x > 0 \) then the upper limit approaches \( \infty \) and \( Q \) approaches 1 and if \( x < 0 \) then the upper limit approaches \( -\infty \) and \( Q \) approaches 0. Thus
\[
1 = c_1 \int_0^\infty e^{-p^2} \, dp + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2
\]
and
\[
0 = c_1 \int_0^{-\infty} e^{-p^2} \, dp + c_2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2.
\]
Thus
\[
c_1 = \frac{1}{\sqrt{\pi}} \quad \text{and} \quad c_2 = \frac{1}{2}.
\]
It follows that
\[
Q(x, t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{4\pi t}} e^{-p^2} \, dp.
\]

**Step 4** Now we know \( Q \) we put \( S = Q_x \). Then \( S \) is also a solution of the diffusion equation, by (b). Given \( \phi \), we define
\[
u(x, t) = \int_{-\infty}^\infty S(x - y, t)\phi(y) \, dy \quad \text{for} \quad t > 0
\]
Then \( u \) is a solution of the diffusion equation by (d).

We have
\[
u(x, t) = \int_{-\infty}^\infty Q_x(x - y, t)\phi(y) \, dy
\]
\[
= -\int_{-\infty}^\infty Q_y(x - y, t)\phi(y) \, dy
\]
\[
= \left[-Q(x - y)\phi(y)\right]_{-\infty}^\infty + \int_{-\infty}^\infty Q(x - y, t)\phi'(y) \, dy.
\]
We assume that the first term is zero; in fact we assume that \( \phi(y) \) is zero for \( y \) large. It follows that

\[
\begin{align*}
    u(x, 0) &= \int_{-\infty}^{\infty} Q(x - y, 0)\phi'(y) \, dy \\
    &= \int_{-\infty}^{\infty} H(x - y)\phi'(y) \, dy \\
    &= \int_{-\infty}^{x} \phi'(y) \, dy \\
    &= \left[ \phi(y) \right]_{-\infty}^{x} \\
    &= \phi(x).
\end{align*}
\]

Now

\[
S = Q_x = \frac{1}{2\sqrt{\pi kt}} e^{-x^2/4kt} \quad \text{for} \quad t > 0.
\]

Thus

\[
u(x, t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) \, dy.
\]