13. Diffusion on the half line

Now we look at diffusion on a half line. We try to solve the Dirichlet boundary problem

$$\begin{aligned} v_t &= kv_{xx} \qquad \text{for} \qquad 0 < x < \infty, \quad 0 < t < \infty \\ v(x,0) &= \phi(x) \qquad \text{for} \qquad t = 0 \\ v(0,t) &= 0 \qquad \text{for} \qquad x = 0. \end{aligned}$$

The solution, if it exists, is unique. Physically it represents the temperature in a very long metal bar, where one end is in a liquid bath, kept at temperature zero.

It is both surprisingly hard and easy to solve this equation. Hard in the sense that one has to be clever to figure out what do; easy in the sense that once one figures this out, it is relatively straightforward.

The trick is to use odd functions; a function ψ is odd if

$$\psi(-x) = -\psi(x).$$

Given ϕ there is a unique function ϕ_{odd} which is the same as ϕ for $x > 0$ (ϕ_{odd} extends ϕ) and which is also odd.

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{if } x > 0 \\ -\phi(-x) & \text{if } x < 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Now we solve the auxiliary problem

$$\begin{aligned} u_t &= ku_{xx} \qquad \text{for} \qquad -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,0) &= \phi_{\text{odd}}(x) \qquad \text{for} \qquad t = 0. \end{aligned}$$

We have a formula for

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\,\mathrm{d}y.$$

Let $v(x,t)$ be the restriction of $u(x,t)$ to the half line $0 < x < \infty$. Note that as u is odd, $v(0,t) = 0$. As derivatives are computed locally, v_t is the restriction of u_t and v_{xx} is the restriction of u_{xx}. Thus v automatically satisfies the diffusion equation. As ϕ is the restriction of $\phi_{\text{odd}}(x)$ it is automatic that $v(x,0) = \phi(x)$.

In fact we can even derive a formula for v. We have

$$u(x,t) = \int_{0}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y - \int_{-\infty}^{0} S(x-y,t)\phi(-y)\,\mathrm{d}y.$$

If we change variable from $-y$ to y in the second integral, we get

$$u(x,t) = \int_0^\infty (S(x-y,t) - S(x+y,t))\phi(y)\,\mathrm{d}y$$

It follows that

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty (e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt})\phi(y)\,\mathrm{d}y$$

Example 13.1. *Find the solution when $\phi(x) = 1$.*

We apply the formula above. Let

$$p = \frac{x-y}{\sqrt{4kt}} \qquad \text{and} \qquad q = \frac{x+y}{\sqrt{4kt}}.$$

We have

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,\mathrm{d}p - \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2}\,\mathrm{d}q \\ &= \left(\frac{1}{2} + \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right) - \left(\frac{1}{2} - \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right) \\ &= \mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right). \end{aligned}$$

In fact one can play the same trick with Neumann boundary conditions:

$$\begin{aligned} w_t &= kw_{xx} \qquad \text{for} \qquad 0 < x < \infty, \quad 0 < t < \infty \\ w(x,0) &= \phi(x) \qquad \text{for} \qquad t = 0 \\ w_x(0,t) &= 0 \qquad \text{for} \qquad x = 0. \end{aligned}$$

The trick is to use even functions; a function ψ is even if

$$\psi(-x) = -\psi(x).$$

Given ϕ there is a unique function ϕ_{odd} which is the same as ϕ for $x > 0$ (ϕ_{even} extends ϕ) and which is also odd.

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{if } x \geq 0 \\ \phi(-x) & \text{if } x < 0. \end{cases}$$

The key point is that the derivative of an even function is odd. We can argue exactly the same way as before. We solve an auxiliary PDE on the whole line with boundary condition ϕ_{even}. The restriction of the solution to this problem is automatically a solution to the original problem. We get the formula

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty (e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt})\phi(y)\,\mathrm{d}y$$

Example 13.2. *Find the solution when $\phi(x) = 1$.*

As above we get

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,\mathrm{d}p + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2}\,\mathrm{d}q \\ &= \left(\frac{1}{2} + \frac{1}{2}\,\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right) + \left(\frac{1}{2} - \frac{1}{2}\,\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right) \\ &= 1. \end{aligned}$$

However, this is not such a hard solution to guess.