22. Inner products and orthogonality

Fix an interval (a,b). We define the **inner product** of two functions f and g on (a,b) as

$$(f,g) = \int_a^b f(x)g(x)\,\mathrm{d}x.$$

The inner product is a real number. We say that f and g are **orthogonal** if the inner product is zero.

The key property of the Fourier sine series, the Fourier cosine series and the Fourier series is that the eigenfunctions are orthogonal, over the intervals $(0,l)$, $(0,l)$, and $(-l,l)$.

In fact this is no coincidence. Suppose that X_1 and X_2 are two eigenfunctions with eigenvalues λ_1 and λ_2, that is,

$$X_1'' = -\lambda_1 X_1 \qquad \text{and} \qquad X_2'' = -\lambda_2 X_2.$$

We have

$$\begin{aligned}(-X_1'X_2 + X_1X_2')' &= -X_1''X_2 - X_1'X_2' + X_1'X_2' + X_1X_2'' \\ &= -X_1''X_2 + X_1X_2''.\end{aligned}$$

If we integrate both sides we get

$$\begin{aligned}\int_a^b (-X_1''X_2 + X_1X_2'')\,\mathrm{d}x &= \int_a^b (-X_1'X_2 + X_1X_2')'\,\mathrm{d}x \\ &= \Big[-X_1'X_2 + X_1X_2'\Big]_a^b.\end{aligned}$$

Now consider the various types of boundary conditions.

Dirichlet: In this case $X_1(a) = X_2(a) = X_1(b) = X_2(b) = 0$. It follows that when we evaluate the last expression we get zero, since every term is zero.

Neumann: In this case $X_1'(a) = X_2'(a) = X_1'(b) = X_2'(b) = 0$. It follows that when we evaluate the last term we get zero, since every term is still zero.

Periodic: In this case $X_j(a) = X_j(b)$ and $X_j'(a) = X_j'(b)$ for every index j. Now we get zero, as the terms at b cancel with the terms at a.

Robin: One can check we still get zero.

Now consider the LHS. As we have eigenfunctions, we get

$$
\begin{aligned}
0 &= \int_a^b (-X_1''X_2 + X_1X_2'')\,\mathrm{d}x \\
&= \int_a^b (-\lambda_1 X_1X_2 + \lambda_2 X_1X_2)\,\mathrm{d}x \\
&= (\lambda_2 - \lambda_1)\int_a^b X_1X_2\,\mathrm{d}x.
\end{aligned}
$$

It follows that if $\lambda_1 \neq \lambda_2$ then X_1 and X_2 are orthogonal.

In short, eigenfunctions with different eigenvalues are automatically orthogonal.

Consider boundary conditions of the form

$$
\begin{aligned}
\alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) &= 0 \\
\alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) &= 0.
\end{aligned}
$$

Here the symbols using Greek letters are real constants. We say that these boundary conditions are **symmetric** if

$$
\Big[f'(x)g(x) - f(x)g'(x) \Big]_a^b = 0
$$

for any pair of functions f and g which satisfy the boundary conditions. Note that all four boundary conditions above are symmetric.

Theorem 22.1. *If the boundary conditions are symmetric then eigenfunctions with different eigenvalues are orthogonal.*

Proof. Suppose that X_1 and X_2 are two eigenfunctions with eigenvalues λ_1 and λ_2. As we have symmetric boundary conditions we have

$$
\Big[X_1'(x)X_2(x) - X_1(x)X_2'(x) \Big]_a^b = 0.
$$

It follows that

$$
\int_a^b (-X_1''X_2 + X_1X_2'')\,\mathrm{d}x = 0.
$$

Arguing as before, this implies that

$$
(\lambda_2 - \lambda_1)\int_a^b X_1X_2\,\mathrm{d}x = 0,
$$

so that if $\lambda_1 \neq \lambda_2$ then X_1 and X_2 are orthogonal. ∎

As a variation on a theme, we can define an inner product of complex valued functions defined on the interval (a, b). As usual this involves complex conjugation.

$$(f, g) = \int_a^b f(x)\overline{g(x)}\,\mathrm{d}x.$$

The symmetric condition now becomes

$$\left[f'(x)\overline{g(x)} - f(x)\overline{g'(x)}\right]_a^b = 0$$

Theorem 22.2. *If the boundary conditions are symmetric then the eigenvalues are all real and we can find real eigenfunctions.*

Proof. Suppose that λ is an eigenvalue. Let X be an eigenfunction with eigenvalue λ, so that $X'' = -\lambda X$. Note that $\overline{X}'' = -\overline{\lambda X}$. As the boundary conditions are real, it follows that $\overline{X}$ is an eigenfunction with eigenvalue $\overline{\lambda}$.

As before this gives

$$\int_a^b (-X''\overline{X} + X\overline{X}'')\,\mathrm{d}x = \left[X\overline{X} - X\overline{X}\right]_a^b = 0.$$

Now the LHS is also equal to

$$\begin{aligned} 0 &= \int_a^b (-X''\overline{X} + X\overline{X}'')\,\mathrm{d}x \\ &= \int_a^b (-\lambda X\overline{X} + \overline{\lambda}X\overline{X})\,\mathrm{d}x \\ &= (\lambda - \overline{\lambda})\int_a^b X\overline{X}\,\mathrm{d}x. \end{aligned}$$

But the integral cannot be zero and so $\lambda = \overline{\lambda}$. It follows that λ is real.

Now write $X = Y + iZ$, where $Y(x)$ and $Z(x)$ are the real and imaginary parts of $X(x)$. Then

$$Y'' + iZ'' = \lambda(Y + iZ).$$

As λ is real, equating the real and imaginary parts, we must have

$$Y'' = \lambda Y \qquad \text{and} \qquad Z'' = \lambda Z.$$

But then Y and Z are two real eigenfunctions with eigenvalue λ. $\square$

There is also a result for negative eigenvalues

Theorem 22.3. *If the boundary conditions are symmetric and*

$$\Big[f'(x)f(x)\Big]_a^b \leq 0$$

for all real functions satisfying the boundary conditions then there are no negative eigenvalues.

There is another interesting result about eigenvalues which is much harder to prove

Theorem 22.4. *If the boundary conditions are symmetric then there are infinitely many eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \dots$$

and their limit is infinity.

It is worth spending a little bit of time talking about convergence. There are three types of convergence.

Definition 22.5. *We say that the sequence of functions $f_1, f_2, \dots$ converges to f on the interval (a, b)*

pointwise: *if given $x \in (a, b)$ we have $\lim_{n\to\infty} f_n(x) = f(x)$.*

uniformly: *if given ϵ we can find n_0 such that $|f_n(x) - f(x)| < \epsilon$ uniformly for all x.*

L^2: *if*

$$\lim_{n\to\infty} \int_a^b (f_n(x) - f(x)^2 \,\mathrm{d}x = 0.$$

The weakest type of convergence is pointwise convergence. The strongest is uniform convergence. L^2-convergence is the most interesting, it says that on average the square of the distance is going to zero.

Roughly speaking Fourier series typically converge both pointwise and in the L^2-sense but not always uniformly.

Theorem 22.6. *If we have symmetric boundary conditions then the Fourier series for $f(x)$ converges uniformly to $f(x)$ provided*

(1) f is $\mathcal{C}^2$, and
(2) f satisfies the boundary conditions.

Theorem 22.7. *The fourier series for f converges to f in the L^2 sense provided*

$$\int_a^b f^2(x) \,\mathrm{d}x$$

is finite.

For example the Fourier sine series

$$\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$$

for $f(x) = 1$ on the interval $(0, 1)$, converges to $f(x)$ pointwise and in the L^2 sense but not uniformly.