22. INNER PRODUCTS AND ORTHOGONALITY

Fix an interval \((a, b)\). We define the **inner product** of two functions \(f\) and \(g\) on \((a, b)\) as

\[
(f, g) = \int_{a}^{b} f(x)g(x) \, dx.
\]

The inner product is a real number. We say that \(f\) and \(g\) are **orthogonal** if the inner product is zero.

The key property of the Fourier sine series, the Fourier cosine series and the Fourier series is that the eigenfunctions are orthogonal, over the intervals \((0, l)\), \((0, l)\), and \((-l, l)\).

In fact this is no coincidence. Suppose that \(X_1\) and \(X_2\) are two eigenfunctions with eigenvalues \(\lambda_1\) and \(\lambda_2\), that is,

\[
X_1'' = -\lambda_1 X_1 \quad \text{and} \quad X_2'' = -\lambda_2 X_2.
\]

We have

\[
\begin{aligned}
(-X_1'X_2 + X_1X_2')' &= -X_1''X_2 - X_1'X_2' + X_1'X_2' + X_1X_2'' \\
&= -X_1''X_2 + X_1X_2''.
\end{aligned}
\]

If we integrate both sides we get

\[
\int_{a}^{b} (-X_1''X_2 + X_1X_2'') \, dx = \int_{a}^{b} (-X_1'X_2 + X_1X_2')' \, dx
\]

\[
= \left[ -X_1'X_2 + X_1X_2' \right]_{a}^{b}.
\]

Now consider the various types of boundary conditions.

**Dirichlet:** In this case \(X_1(a) = X_2(a) = X_1(b) = X_2(b) = 0\). It follows that when we evaluate the last expression we get zero, since every term is zero.

**Neumann:** In this case \(X_1'(a) = X_2'(a) = X_1'(b) = X_2'(b) = 0\). It follows that when we evaluate the last term we get zero, since every term is still zero.

**Periodic:** In this case \(X_j(a) = X_j(b)\) and \(X_j'(a) = X_j'(b)\) for every index \(j\). Now we get zero, as the terms at \(b\) cancel with the terms at \(a\).

**Robin:** One can check we still get zero.
Now consider the LHS. As we have eigenfunctions, we get

\[0 = \int_a^b (-X_1''X_2 + X_1X_2'') \, dx\]

\[= \int_a^b (-\lambda_1 X_1X_2 + \lambda_2 X_1X_2) \, dx\]

\[= (\lambda_2 - \lambda_1) \int_a^b X_1X_2 \, dx.\]

It follows that if \(\lambda_1 \neq \lambda_2\) then \(X_1\) and \(X_2\) are orthogonal.

In short, eigenfunctions with different eigenvalues are automatically orthogonal.

Consider boundary conditions of the form

\[\alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0\]

\[\alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0.\]

Here the symbols using Greek letters are real constants. We say that these boundary conditions are symmetric if

\[\left[f'(x)g(x) - f(x)g'(x)\right]_a^b = 0\]

for any pair of functions \(f\) and \(g\) which satisfy the boundary conditions. Note that all four boundary conditions above are symmetric.

**Theorem 22.1.** If the boundary conditions are symmetric then eigenfunctions with different eigenvalues are orthogonal.

**Proof.** Suppose that \(X_1\) and \(X_2\) are two eigenfunctions with eigenvalues \(\lambda_1\) and \(\lambda_2\). As we have symmetric boundary conditions we have

\[\left[X_1'(x)X_2(x) - X_1(x)X_2'(x)\right]^b_a = 0.\]

It follows that

\[\int_a^b (-X_1''X_2 + X_1X_2'') \, dx = 0.\]

Arguing as before, this implies that

\[(\lambda_2 - \lambda_1) \int_a^b X_1X_2 \, dx = 0,\]

so that if \(\lambda_1 \neq \lambda_2\) then \(X_1\) and \(X_2\) are orthogonal. 

\[\square\]
As a variation on a theme, we can define an inner product of complex valued functions defined on the interval \((a, b)\). As usual this involves complex conjugation.

\[(f, g) = \int_a^b f(x)\overline{g(x)} \, dx.\]

The symmetric condition now becomes

\[\left[ f'(x)\overline{g(x)} - f(x)\overline{g'(x)} \right]_a^b = 0\]

**Theorem 22.2.** If the boundary conditions are symmetric then the eigenvalues are all real and we can find real eigenfunctions.

*Proof.* Suppose that \(\lambda\) is an eigenvalue. Let \(X\) be an eigenfunction with eigenvalue \(\lambda\), so that \(X'' = -\lambda X\). Note that \(\overline{X''} = -\overline{\lambda X}\). As the boundary conditions are real, it follows that \(\overline{X}\) is an eigenfunction with eigenvalue \(\overline{\lambda}\).

As before this gives

\[\int_a^b (-X''\overline{X} + X\overline{X''}) \, dx = \left[ X\overline{X} - X\overline{X} \right]_a^b = 0.\]

Now the LHS is also equal to

\[0 = \int_a^b (-X''\overline{X} + X\overline{X''}) \, dx = \int_a^b (-\lambda X\overline{X} + \lambda X\overline{X}) \, dx = (\lambda - \overline{\lambda}) \int_a^b X\overline{X} \, dx.\]

But the integral cannot be zero and so \(\lambda = \overline{\lambda}\). It follows that \(\lambda\) is real.

Now write \(X = Y + iZ\), where \(Y(x)\) and \(Z(x)\) are the real and imaginary parts of \(X(x)\). Then

\[Y'' + iZ'' = \lambda(Y + iZ).\]

As \(\lambda\) is real, equating the real and imaginary parts, we must have

\[Y'' = \lambda Y \quad \text{and} \quad Z'' = \lambda Z.\]

But then \(Y\) and \(Z\) are two real eigenfunctions with eigenvalue \(\lambda\). \(\Box\)

There is also a result for negative eigenvalues
Theorem 22.3. If the boundary conditions are symmetric and
\[ \left[ f'(x)f(x) \right]_a^b \leq 0 \]
for all real functions satisfying the boundary conditions then there are no negative eigenvalues.

There is another interesting result about eigenvalues which is much harder to prove

Theorem 22.4. If the boundary conditions are symmetric then there are infinitely many eigenvalues
\[ \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \]
and their limit is infinity.

It is worth spending a little bit of time talking about convergence. There are three types of convergence.

Definition 22.5. We say that the sequence of functions \( f_1, f_2, \ldots \) converges to \( f \) on the interval \((a,b)\)

pointwise: if given \( x \in (a,b) \) we have \( \lim_{n \to \infty} f_n(x) = f(x) \).

uniformly: if given \( \epsilon \) we can find \( n_0 \) such that \( |f_n(x) - f(x)| < \epsilon \) uniformly for all \( x \).

\( L^2 \): if
\[ \lim_{n \to \infty} \int_a^b (f_n(x) - f(x))^2 \, dx = 0. \]

The weakest type of convergence is pointwise convergence. The strongest is uniform convergence. \( L^2 \)-convergence is the most interesting, it says that on average the square of the distance is going to zero.

Roughly speaking Fourier series typically converge both pointwise and in the \( L^2 \)-sense but not always uniformly.

Theorem 22.6. If we have symmetric boundary conditions then the Fourier series for \( f(x) \) converges uniformly to \( f(x) \) provided

(1) \( f \) is \( C^2 \), and
(2) \( f \) satisfies the boundary conditions.

Theorem 22.7. The fourier series for \( f \) converges to \( f \) in the \( L^2 \) sense provided
\[ \int_a^b f^2(x) \, dx \]
is finite.
For example the Fourier sine series
\[ \frac{4}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \ldots \right) \]
for \( f(x) = 1 \) on the interval \((0, 1)\), converges to \( f(x) \) pointwise and in the \( L^2 \) sense but not uniformly.