23. Harmonic functions

Recall Laplace's equation

$$\begin{aligned}\Delta u &= u_{xx} = 0\\ \Delta u &= u_{xx} + u_{yy} = 0\\ \Delta u &= u_{xx} + u_{yy} + u_{zz} = 0.\end{aligned}$$

Solutions to Laplace's equation are called harmonic functions.

The inhomogeneous version of Laplace's equation

$$\Delta u = f$$

is called the **Poisson equation**.

Harmonic functions are Laplace's equation turn up in many different places in mathematics and physics.

Harmonic functions in one variable are easy to describe. The general solution of

$$u_{xx} = 0$$

is $u(x) = ax + b$, for constants a and b.

Maximum principle Let D be a connected and bounded open set in $\mathbb{R}^2$. Let $u(x, y)$ be a harmonic function on D that has a continuous extension to the boundary ∂D of D.

Then the maximum (and minimum) of u are attained on the boundary and if they are attained anywhere else than u is constant.

Euivalently, there are two points (x_m, y_m) and (x_M, y_M) on the boundary such that

$$u(x_m, y_m) \leq u(x, y) \leq u(x_M, y_M)$$

for every point of D and if we have equality then u is constant.

The idea of the proof is as follows. At a maximum point of u in D we must have $u_{xx} \leq 0$ and $u_{yy} \leq 0$. Most of the time one of these inequalities is strict and so

$$0 = u_{xx} + u_{yy} < 0,$$

which is not possible. The only reason this is not a full proof is that sometimes both $u_{xx} = u_{yy} = 0$.

As before, to fix this, simply perturb away from zero. Let $\epsilon > 0$ and let

$$v(x, y) = u(x, y) + \epsilon(x^2 + y^2).$$

Then

$$\begin{aligned}\Delta v &= \Delta u + \epsilon\Delta(x^2 + y^2)\\ &= 4\epsilon\\ &> 0.\end{aligned}$$

Thus v has no maximum in D.

As v is continuous it must achieve its maximum at $(x_0, y_0) \in \partial D$. For any $(x, y) \in D$ we have

$$\begin{aligned} u(x, y) &\leq v(x, y) \\ &\leq v(x_0, y_0) \\ &= u(x_0, y_0) + \epsilon(x_0^2 + y_0^2) \\ &\max_{(x_1,y_1)\in\partial D} u(x_1, y_1) + \epsilon l^2, \end{aligned}$$

where l is the greatest distance of the boundary to the origin.

As this is true for every ϵ, we must have

$$u(x, y) \leq \max_{(x_1,y_1)\in\partial D} u(x_1, y_1).$$

For the minimum, just apply the maximum principle to $-u$ which is harmonic.

As usual, we can use this to prove uniqueness of solutions. For example, consider the Dirichlet problem

$$\Delta u = f \text{ on } D \qquad \text{and} \qquad u = h \text{ on } \partial D.$$

Suppose there were two solutions u and v. Consider the difference $w = v - u$. Then w is harmonic and zero on the boundary. By the maxmimum principle, $w \leq 0$ and by the minimum principle $w \geq 0$. But then $w = 0$ so that $u = v$ and the solution is unique.

We now turn to trying to solve Laplace's equation. The answer depends heavily on the geometry of D.

We use the following rubric

(i) Look for separated solutions of the PDE.
(ii) Impose the homogeneous boundary conditions.
(iii) Sum the series.
(iv) Impose the inhomogeneous initial and boundary conditions.

We carry this out for a rectangle,

$$(0, a) \times (0, b).$$

Let's suppose that we have one of each of the three standard boundary conditions on each side.

$$u = j(y) \qquad u_x = h(y) \qquad u_y + u = h(x) \qquad \text{and} \qquad u = g(x).$$

As the Laplacian is linear we can break up the solution we are looking for into four parts

$$u = u_1 + u_2 + u_3 + u_4.$$

We can think of u_1 as being a harmonic function which satisfies the boundary condition for j, where we set all three other functions equal to zero, and so on around the sides of the rectangle.

So let's deal with the case that only g is non-zero.

We now separate variables

$$u(x, y) = X(x)Y(y).$$

Then

$$X''Y + XY'' = 0,$$

so that

$$\frac{X''}{X} = -\frac{Y''}{Y}.$$

As usual this implies both sides are constant. It follows that

$$X'' = -\lambda X \qquad \text{and} \qquad Y'' = \lambda Y,$$

for some constant λ. The boundary conditions imply that $X(0) = X'(a) = 0$, so that

$$\lambda_n = \left(n + \frac{1}{2}\right)^2,$$

for $n = 1, 2, \ldots$. Thus

$$\beta_n = n + \frac{1}{2}$$

The eigenfunctions for X are

$$X_n(x) = \sin\frac{(n + \frac{1}{2})\pi x}{a}.$$

The boundary conditions for Y are

$$Y'(0) + Y(0) = 0.$$

The solutions of the ODE are exponentials and it is best to write them as

$$Y_n(y) = A\cosh\beta_n y + B\sinh\beta_n y.$$

The boundary conditions reduce to

$$A + \beta_n B = 0.$$

It we take $B = -1$ then $A = \beta_n$ and we get

$$Y_n(y) = \beta_n\cosh\beta_n y - \sinh\beta_n y.$$

Now we sum

$$u(x, y) = \sum_n A_n \sin\beta_n x(\beta_n\cosh\beta_n y - \sinh\beta_n y).$$

This satisfies the homogeneous boundary conditions. We now tune the coefficients $A_1, A_2, \ldots$ to fit the boundary condition

$$u(x, b) = g(x).$$

This gives

$$g(x) = \sum_n A_n(\beta_n \cosh \beta_n b - \sinh \beta_n b) \sin \beta_n x.$$

Notice that this is simply a Fourier series in the eigenfunctions $\sin \beta_n x$. Thus the coefficients are determined by the usual formula

$$A_n = \frac{2}{a}(\beta_n \cosh \beta_n b - \sinh \beta_n b)^{-1} \int_0^a g(x) \sin \beta_n x \, \mathrm{d}x.$$