FIRST MIDTERM
MATH 110A, UCSD, AUTUMN 18

You have 80 minutes.
There are 5 problems, and the total number of points is 65. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name: __________________________

Signature: _________________________

Student ID #: ______________________

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1. (15pts) (i) Give the definition of a linear operator.

An operator \( \mathcal{L} \) is linear if
\[
\mathcal{L}(u + v) = \mathcal{L} u + \mathcal{L} v \quad \text{and} \quad \mathcal{L}(cu) = c \mathcal{L} u.
\]

(ii) Write down the PDE describing a vibrating string.

If \( u(x, t) \) is the vertical displacement of the string from equilibrium then
\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{where} \quad c = \sqrt{\frac{T}{\rho}},
\]
where \( T \) is the tension and \( \rho \) is the density.

(iii) Write down the Laplacian in three dimensions.

\[
\Delta u = u_{xx} + u_{yy} + u_{zz}.
\]
2. (15pts) (i) Find the general solution of the PDE

\[ u_{xy} = 0, \]

where \( u(x, y) \) depends on \( x \) and \( y \).

Integrating with respect to \( x \) gives

\[ u_y = f(y) \]

where \( f \) is an arbitrary function of one variable. Integrating with respect to \( y \) gives

\[ u(x, y) = F(y) + G(x), \]

where \( G \) is an arbitrary function of one variable and \( F' = f \) so that \( F \)

is also an arbitrary function of one variable.

(ii) Find the solution of the PDE

\[ 4u_t - 5u_x = 0, \]

subject to the auxiliary condition \( u = \tan x \) when \( t = 0 \), where \( u(x, t) \)
depends on \( x \) and \( t \).

The general solution of a constant coefficient first order linear PDE is

\[ u(x, t) = f(4x + 5t), \]

where \( f \) is an arbitrary function of one variable. If we impose the auxiliary condition then we get

\[ \tan x = u(x, 0) \]

\[ = f(4x). \]

If we put \( w = 4x \) then \( x = w/4 \) and

\[ f(w) = \tan w/4. \]

Thus the solution to the PDE is

\[ u(x, t) = \tan \left( \frac{4x + 5t}{4} \right). \]
(iii) Solve the equation $yu_x + xu_y = 0$ with $u(x,0) = e^{-x^2}$.

The tangent vector to the characteristic curve at $(x, y)$ is $(y, x)$. It follows that the characteristic curve satisfies the ODE

$$\frac{dy}{dx} = \frac{x}{y}.$$  

Separating the variables, we see that the characteristic curves have equations

$$y^2 = x^2 + c.$$  

It follows that the general solution to the original PDE is

$$u(x, y) = f(y^2 - x^2),$$

where $f$ is an arbitrary function of one variable. If we impose the auxiliary condition we get

$$e^{-x^2} = u(x, 0) = f(-x^2).$$

If we let $w = x^2$ then we get

$$f(w) = e^w.$$  

Therefore the solution to the original problem is

$$u(x, y) = e^{y^2-x^2}.$$
3. (10pts) Consider the ODE
\[
\frac{d^2 u}{dx^2} + u = 0
\]
subject to the auxiliary conditions \(u(0) = 0\) and \(u(L) = 0\).
When is the solution unique?

The general solution to the ODE is
\[
u(x) = a \cos x + b \sin x.
\]
The condition that \(u(0) = 0\) implies that \(a = 0\). The condition that \(u(L) = 0\) implies that \(b \sin(L) = 0\).
This implies that \(b = 0\) unless \(\sin(L) = 0\). Now \(\sin(L) = 0\) if and only if \(L\) is an integer multiple of \(\pi\).
Thus the solution to the PDE is unique if and only if \(L\) is not an integer multiple of \(\pi\).
4. (10pts) Find the regions in the \((x, y)\)-plane where the equation
\[
yu_{xx} - 2u_{xy} + xu_{yy} = 0
\]
is elliptic, hyperbolic and parabolic.

We have
\[
a_{11} = y \quad a_{12} = -1 \quad \text{and} \quad a_{22} = x.
\]
It follows that
\[
\mathcal{D} = a_{12}^2 - a_{11}a_{22}
= 1^2 - xy
= 1 - xy.
\]
Thus the PDE is parabolic on the two branches of the hyperbola \(xy = 1\), it is elliptic in the two convex regions belonging to the first and third quadrant, where \(xy > 1\), and it is hyperbolic in the remaining region \(xy < 1\).
5. (15pts) A rod occupying the interval $0 \leq x \leq l$ is subject to the heat source $f(x) = 0$ for $0 < x < l/2$ and $f(x) = H$ for $l/2 < x < l$. The rod has physical constants $c = \rho = \kappa = 1$ and its end are kept at zero temperature.

(i) Find the steady state of the temperature of the rod.

With the assumption on the constants, and the fact that when we have steady state, $u_t = 0$, the heat equation reduces to

$$u_{xx} = -H(x).$$

We solve this equation on both intervals. Over the interval $0 < x < l/2$ we have the equation

$$u_{xx} = 0$$

and this has solution

$$u(x, t) = ax + b,$$

where $a$ and $b$ are constants to be determined.

Over the interval $l/2 < x < l$ we have the equation

$$u_{xx} = -H.$$

The general solution is

$$u(x, t) = -Hx^2/2 + cx + d,$$

where $c$ and $d$ are constants to be determined.

There are four boundary conditions, what happens at the two endpoints, the condition that both solutions are equal at $l/2$ and the condition that the heat flow matches at $l/2$.

The two endpoints give

$$b = 0 \quad \text{and} \quad -\frac{Hl^2}{2} + cl + d = 0.$$

Thus

$$d = \frac{Hl^2}{2} - cl.$$

Matching $u$ at $x = l/2$ gives

$$\frac{al}{2} = -\frac{Hl^2}{8} + \frac{cl}{2} + d.$$

Matching $u_x$ at $x = l/2$ gives

$$a = -\frac{Hl}{2} + c.$$
Substituting for $a$ and $d$ gives an equation for $c$:

\[
\frac{cl}{2} - \frac{Hl^2}{4} = -\frac{Hl^2}{8} + \frac{cl}{2} + \frac{Hl^2}{2} - cl.
\]

Thus

\[cl = \frac{5Hl^2}{8}.
\]

It follows that

\[c = \frac{5Hl}{8}.
\]

Thus

\[a = \frac{Hl}{8} \quad \text{and} \quad d = -\frac{Hl^2}{8}.
\]

The solution is

\[
u(x, t) = \begin{cases} 
\frac{Hlx}{8} & 0 \leq x \leq l/2 \\
-Hx^2/2 + 5Hlx/8 - Hl^2/8 & l/2 \leq x.
\end{cases}
\]

(ii) Which is the hottest point?

Over the interval $0 \leq x \leq l/2$ the maximum is at $x = l/2$ and the maximum is

\[\frac{Hl^2}{16}.
\]

Over the interval $l/2 \leq x \leq l$ the maximum is at $x = 5l/8$ and the maximum is

\[\frac{9Hl^2}{128}.
\]

and this is the hottest temperature, so that $x = 5l/8$ is the hottest point.
Bonus Challenge Problems
6. (10pts) Consider heat flow in a large spherical ball where the temperature only depends on $t$ and on the distance $r$ to the centre of the ball. Derive the equation

$$u_t = k(u_{rr} + 2u_r/r).$$

We assume that $\kappa$ is constant, so that the PDE for the heat equation reduces to

$$u_t = k\Delta u.$$  

In spherical coordinates we have $(\rho, \theta, \phi)$ where $\rho$ is the distance to the origin, $\theta$ is the same angle as in cylindrical coordinates and $\phi$ is the angle from the $z$-axis. We have

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

By assumption $u_\theta = 0$ and $u_\phi = 0$. We rename $\rho = r$. The chain rule gives

$$u_{xx} = (u_x)_x$$

$$= (xr^{-1}u_r)_x$$

$$= r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-2}u_{rr}.$$  

Similarly

$$u_{yy} = r^{-1}u_r - y^2r^{-3}u_r + y^2r^{-2}u_{rr}$$

and

$$u_{zz} = r^{-1}u_r - z^2r^{-3}u_r + z^2r^{-2}u_{rr}.$$  

Therefore

$$\Delta u = u_{xx} + u_{yy} + u_{zz}$$

$$= r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-3}u_{rr} + r^{-1}u_r - y^2r^{-2}u_r + y^2r^{-2}u_{rr} + r^{-1}u_r - z^2r^{-3}u_r + z^2r^{-2}u_{rr}$$

$$= 3r^{-1}u_r - (x^2 + y^2 + z^2)r^{-3}u_r + (x^2 + y^2 + z^2)r^{-2}u_{rr}$$

$$= u_{rr} + 2r^{-1}u_r.$$  

Thus the heat equation reduces to

$$u_t = k(u_{rr} + 2u_r/r).$$
7. (10pts) Two homogeneous rods have the same cross section, specific heat \( c \) and density \( \rho \) but different heat conductivities \( \kappa_1 \) and \( \kappa_2 \) and lengths \( L_1 \) and \( L_2 \). Let

\[
  k_j = \frac{\kappa_j}{c\rho} \quad \text{for} \quad j = 1, 2
\]

be their diffusion constants. They are welded together so that the temperature \( u \) and the heat flux \( \kappa u_x \) at the weld are continuous. The left-hand rod has its left end maintained at temperature zero whilst the right-hand rod has its right end maintained at temperature \( T \).

Find the equilibrium temperature distribution in the composite rod.

At equilibrium we have \( u_t = 0 \) and so the heat equation reads

\[
u_{xx} = 0.
\]

Solving on the interval \( 0 \leq x \leq L_1 \) we have

\[
u(x, t) = ax + b
\]

and on the interval \( L_1 \leq x \leq L_1 + L_2 \) we have

\[
u(x, t) = cx + d.
\]

There are four boundary conditions, what happens at the two endpoints, and the condition that both \( u \) and the flux are continuous. As the temperature is zero at 0 we have

\[
b = u(0, t) = 0.
\]

As the temperature is \( T \) at \( x = L_1 + L_2 \) have

\[
c(L_1 + L_2) + d = T.
\]

As \( u \) is continuous at \( x = L_1 \) we have

\[
aL_1 = cL_1 + d
\]

As the flux is continuous at \( x = L_1 \) we must have

\[
\kappa_1 a = \kappa_2 c.
\]

From the second equation we get

\[
d = T - c(L_1 + L_2).
\]

Plugging this into the third equation gives

\[
T = aL_1 + cL_2.
\]

Multiplying through by \( \kappa_2 \) and using the fourth equation gives

\[
\kappa_2 T = \kappa_2 aL_1 + \kappa_1 aL_2 = a(\kappa_2 L_1 + \kappa_1 L_2).
\]

It follows that
\[ a = \frac{\kappa_2 T}{\kappa_2 L_1 + \kappa_1 L_2}. \]
From there we get
\[ c = \frac{\kappa_1 T}{\kappa_2 L_1 + \kappa_1 L_2}. \]
It follows that
\[ d = \frac{(\kappa_2 - \kappa_1) L_1 T}{\kappa_2 L_1 + \kappa_1 L_2}. \]
Thus
\[
   u(x, t) = \begin{cases} 
   \frac{\kappa_2 T x}{\kappa_2 L_1 + \kappa_1 L_2} & \text{for } 0 < x < L_1 \\
   \frac{(\kappa_2 - \kappa_1) L_1 T}{\kappa_2 L_1 + \kappa_1 L_2} + \frac{\kappa_1 T x}{\kappa_2 L_1 + \kappa_1 L_2} & \text{for } L_1 < x < L_1 + L_2.
   \end{cases}
\]