FIRST MIDTERM
MATH 110A, UCSD, AUTUMN 18

You have 80 minutes.

There are 5 problems, and the total number of points is 65. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:________________________________

Signature:________________________________

Student ID #:________________________________

Problem	Points	Score
1	15	
2	15	
3	10	
4	10	
5	15	
6	10	
7	10	
Total	65	

1. (15pts) (i) *Give the definition of a linear operator.*

An operator $\mathscr{L}$ is linear if

$$\mathscr{L}(u+v) = \mathscr{L}\,u + \mathscr{L}\,v \qquad \text{and} \qquad \mathscr{L}(cu) = c\,\mathscr{L}\,u.$$

(ii) *Write down the PDE describing a vibrating string.*

If $u(x,t)$ is the vertical displacement of the string from equilibrium then

$$u_{tt} = c^2 u_{xx} \qquad \text{where} \qquad c = \sqrt{\frac{T}{\rho}},$$

where T is the tension and ρ is the density.

(iii) *Write down the Laplacian in three dimensions.*

$$\Delta u = u_{xx} + u_{yy} + u_{zz}.$$

2. (15pts) (i) *Find the general solution of the PDE*

$$u_{xy} = 0,$$

where $u(x, y)$ depends on x and y.

Integrating with respect to x gives

$$u_y = f(y)$$

where f is an arbitrary function of one variable. Integrating with respect to y gives

$$u(x, y) = F(y) + G(x),$$

where G is an arbitrary function of one variable and $F' = f$ so that F is also an arbitrary function of one variable.

(ii) *Find the solution of the PDE*

$$4u_t - 5u_x = 0,$$

subject to the auxiliary condition $u = \tan x$ when $t = 0$, where $u(x, t)$ depends on x and t.

The general solution of a constant coefficient first order linear PDE is

$$u(x, t) = f(4x + 5t),$$

where f is an arbitrary function of one variable. If we impose the auxiliary condition then we get

$$\begin{aligned} \tan x &= u(x, 0) \\ &= f(4x). \end{aligned}$$

If we put $w = 4x$ then $x = w/4$ and

$$f(w) = \tan w/4.$$

Thus the solution to the PDE is

$$u(x, t) = \tan\left(\frac{4x + 5t}{4}\right).$$

(iii) *Solve the equation* $yu_x + xu_y = 0$ *with* $u(x, 0) = e^{-x^2}$.

The tangent vector to the characteristic curve at (x, y) is (y, x). It follows that the characteristic curve satisfies the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}.$$

Separating the variables, we see that the characteristic curves have equations

$$y^2 = x^2 + c.$$

It follows that the general solution to the original PDE is

$$u(x, y) = f(y^2 - x^2),$$

where f is an arbitrary function of one variable. If we impose the auxiliary condition we get

$$\begin{aligned} e^{-x^2} &= u(x, 0) \\ &= f(-x^2). \end{aligned}$$

If we let $w = x^2$ then we get

$$f(w) = e^w.$$

Therefore the solution to the original problem is

$$u(x, y) = e^{y^2 - x^2}.$$

3. (10pts) *Consider the ODE*

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + u = 0$$

subject to the auxiliary conditions $u(0) = 0$ *and* $u(L) = 0$.
When is the solution unique?

The general solution to the ODE is

$$u(x) = a\cos x + b\sin x.$$

The condition that $u(0) = 0$ implies that $a = 0$. The condition that $u(L) = 0$ implies that

$$b\sin(L) = 0.$$

This implies that $b = 0$ unless $\sin(L) = 0$. Now $\sin(L) = 0$ if and only if L is an integer multiple of π.
Thus the solution to the PDE is unique if and only if L is not an integer multiple of π.

4. (10pts) *Find the regions in the (x, y)-plane where the equation*

$$yu_{xx} - 2u_{xy} + xu_{yy} = 0$$

is elliptic, hyperbolic and parabolic.

We have

$$a_{11} = y \qquad a_{12} = -1 \qquad \text{and} \qquad a_{22} = x.$$

It follows that

$$\begin{aligned} \mathscr{D} &= a_{12}^2 - a_{11}a_{22} \\ &= 1^2 - xy \\ &= 1 - xy. \end{aligned}$$

Thus the PDE is parabolic on the two branches of the hyperbola $xy = 1$, it is elliptic in the two convex regions belonging to the first and third quadrant, where $xy > 1$, and it is hyperbolic in the remaining region $xy < 1$.

5. (15pts) *A rod occupying the interval* $0 \le x \le l$ *is subject to the heat source* $f(x) = 0$ *for* $0 < x < l/2$ *and* $f(x) = H$ *for* $l/2 < x < l$. *The rod has physical constants* $c = \rho = \kappa = 1$ *and its end are kept at zero temperature.*
(i) *Find the steady state of the temperature of the rod.*

With the assumption on the constants, and the fact that when we have steady state, $u_t = 0$, the heat equation reduces to

$$u_{xx} = -H(x).$$

We solve this equation on both intervals. Over the interval $0 < x < l/2$ we have the equation

$$u_{xx} = 0$$

and this has solution

$$u(x,t) = ax + b,$$

where a and b are constants to be determined.
Over the interval $l/2 < x < l$ we have the equation

$$u_{xx} = -H.$$

The general solution is

$$u(x,t) = -Hx^2/2 + cx + d,$$

where c and d are constants to be determined.
There are four boundary conditions, what happens at the two endpoints, the condition that both solutions are equal at $l/2$ and the condition that the heat flow matches at $l/2$.
The two endpoints give

$$b = 0 \qquad \text{and} \qquad -\frac{Hl^2}{2} + cl + d = 0.$$

Thus

$$d = \frac{Hl^2}{2} - cl.$$

Matching u at $x = l/2$ gives

$$\frac{al}{2} = -\frac{Hl^2}{8} + \frac{cl}{2} + d.$$

Matching u_x at $x = l/2$ gives

$$a = -\frac{Hl}{2} + c.$$

Substituting for a and d gives an equation for c:

$$\frac{cl}{2} - \frac{Hl^2}{4} = -\frac{Hl^2}{8} + \frac{cl}{2} + \frac{Hl^2}{2} - cl.$$

Thus

$$cl = \frac{5Hl^2}{8}.$$

It follows that

$$c = \frac{5Hl}{8}.$$

Thus

$$a = \frac{Hl}{8} \qquad \text{and} \qquad d = -\frac{Hl^2}{8}.$$

The solution is

$$u(x,t) = \begin{cases} \frac{Hlx}{8} & 0 \le x \le l/2 \\ -\frac{Hx^2}{2} + \frac{5Hlx}{8} - \frac{Hl^2}{8} & l/2 \le x. \end{cases}$$

(ii) *Which is the hottest point?*

Over the interval $0 \le x \le l/2$ the maximum is at $x = l/2$ and the maximum is

$$\frac{Hl^2}{16}.$$

Over the interval $l/2 \le x \le l$ the maximum is at $x = 5l/8$ and the maximum is

$$\frac{9Hl^2}{128}.$$

and this is the hottest temperature, so that $x = 5l/8$ is the hottest point.

Bonus Challenge Problems

6. (10pts) *Consider heat flow in a large spherical ball where the temperature only depends on t and on the distance r to the centre of the ball. Derive the equation*

$$u_t = k(u_{rr} + 2u_r/r).$$

We assume that κ is constant, so that the PDE for the heat equation reduces to

$$u_t = k\Delta u.$$

In spherical coordinates we have (ρ, θ, ϕ) where ρ is the distance to the origin, θ is the same angle as in cylindrical coordinates and ϕ is the angle from the z-axis. We have

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

By asumption $u_\theta = 0$ and $u_\phi = 0$. We rename $\rho = r$. The chain rule gives

$$\begin{aligned} u_{xx} &= (u_x)_x \\ &= (xr^{-1}u_r)_x \\ &= r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-2}u_{rr}. \end{aligned}$$

Similarly

$$u_{yy} = r^{-1}u_r - y^2r^{-3}u_r + y^2r^{-2}u_{rr} \qquad \text{and} \qquad u_{zz} = r^{-1}u_r - z^2r^{-3}u_r + z^2r^{-2}u_{rr}.$$

Therefore

$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} + u_{zz} \\ &= r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-3}u_{rr} + r^{-1}u_r - y^2r^{-2}u_r + y^2r^{-2}u_{rr} + r^{-1}u_r - z^2r^{-3}u_r + z^2r^{-2}u_{rr} \\ &= 3r^{-1}u_r - (x^2 + y^2 + z^2)r^{-3}u_r + (x^2 + y^2 + z^2)r^{-2}u_{rr} \\ &= u_{rr} + 2r^{-1}u_r. \end{aligned}$$

Thus the heat equation reduces to

$$u_t = k(u_{rr} + 2u_r/r).$$

7. (10pts) *Two homogeneous rods have the same cross section, specific heat c and density ρ but different heat conductivities κ_1 and κ_2 and lengths L_1 and L_2. Let*

$$k_j = \frac{\kappa_j}{c\rho} \qquad \text{for} \qquad j = 1, 2$$

be their diffusion constants. They are welded together so that the temperature u and the heat flux κu_x at the weld are continuous. The left-hand rod has it left end maintained at temperature zero whilst the right-hand rod has it right end maintained at temperature T.
Find the equilibrium temperature distribution in the composite rod.

At equilibrium we have $u_t = 0$ and so the heat equation reads

$$u_{xx} = 0.$$

Solving on the interval $0 \leq x \leq L_1$ we have

$$u(x, t) = ax + b$$

and on the interval $L_1 \leq x \leq L_1 + L_2$ we have

$$u(x, t) = cx + d.$$

There are four boundary conditions, what happens at the two end-points, and the condition that both u and the flux are continuous. As the temperature is zero at 0 we have

$$b = u(0, t) = 0.$$

As the temperature is T at $x = L_1 + L_2$ have

$$c(L_1 + L_2) + d = T.$$

As u is continuous at $x = L_1$ we have

$$aL_1 = cL_1 + d$$

As the flux is continuous at $x = L_1$ we must have

$$\kappa_1 a = \kappa_2 c.$$

From the second equation we get

$$d = T - c(L_1 + L_2).$$

Plugging this into the third equation gives

$$T = aL_1 + cL_2.$$

Multiplying through by κ_2 and using the fourth equation gives

$$\kappa_2 T = \kappa_2 a L_1 + \kappa_1 a L_2 = a(\kappa_2 L_1 + \kappa_1 L_2).$$

It follows that

$$a = \frac{\kappa_2 T}{\kappa_2 L_1 + \kappa_1 L_2}.$$

From there we get

$$c = \frac{\kappa_1 T}{\kappa_2 L_1 + \kappa_1 L_2}.$$

It follows that

$$d = \frac{(\kappa_2 - \kappa_1) L_1 T}{\kappa_2 L_1 + \kappa_1 L_2}.$$

Thus

$$u(x,t) = \begin{cases} \frac{\kappa_2 T x}{\kappa_2 L_1 + \kappa_1 L_2} & \text{for } 0 < x < L_1 \\ \frac{\kappa_1 T x}{\kappa_2 L_1 + \kappa_1 L_2} + \frac{(\kappa_2 - \kappa_1) L_1 T}{\kappa_2 L_1 + \kappa_1 L_2} & \text{for } L_1 < x < L_1 + L_2. \end{cases}$$