MODEL ANSWERS TO THE FIRST HOMEWORK

1.1.2 (a) linear;

$$\begin{aligned}\mathscr{L}(u+v) &= (u+v)_x + x(u+v)_y \\ &= u_x + v_x + xu_y + xv_y \\ &= \mathscr{L}\,u + \mathscr{L}\,v\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}(cu) &= (cu)_x + x(cu)_x \\ &= cu_x + cxu_x \\ &= c\,\mathscr{L}\,u.\end{aligned}$$

(b) Not linear;

$$\begin{aligned}\mathscr{L}(2u) &= (2u)_x + (2u)(2u)_y \\ &= 2u_x + 4uu_y \\ &\neq 2u_x + 2uu_y \\ &= 2\,\mathscr{L}\,u.\end{aligned}$$

(d) Not linear;

$$\begin{aligned}\mathscr{L}(2u) &= (2u)_x + (2u)_y + 1 \\ &\neq 2u_x + 2u_y + 2 \\ &= 2\,\mathscr{L}\,u.\end{aligned}$$

1.1.3 (a) The order is two, due to the term u_{xx}; it is linear inhomogeneous, as we can put the equation into the form

$$u_t - u_{xx} = -1$$

and the operator

$$\mathscr{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

is linear;

$$\begin{aligned}\mathscr{L}(u+v) &= (u+v)_t - (u+v)_{xx} \\ &= u_t + v_t - u_{xx} - v_{xx} \\ &= \mathscr{L}\,u + \mathscr{L}\,v\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}(cu) &= (cu)_t - (cu)_{xx}\\ &= cu_t - cu_{xx}\\ &= c\mathscr{L}u.\end{aligned}$$

(c) The order is three, due to the term u_{xxt}; it is nonlinear, as there is no term that does not depend on u and the operator

$$f(u) = u_t - u_{xxt} + uu_x$$

is not linear;

$$\begin{aligned}f(2u) &= (2u)_t - (2u)_{xxt} + (2u)(2u)_x\\ &= 2u_t - 2u_{xxt} + 4uu_x\\ &\neq 2u_t - 2u_{xxt} + 2uu_x\\ &= 2f(u).\end{aligned}$$

(e) The order is two, due to the term u_{xx}; it is linear homogeneous, as the operator

$$\mathscr{L} = i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{u}{x}$$

is linear;

$$\begin{aligned}\mathscr{L}(u+v) &= i(u+v)_t - (u+v)_{xx} + \frac{(u+v)}{x}\\ &= iu_t + iv_t - v_{xx} - u_{xx} + \frac{u}{x} + \frac{v}{x}\\ &= \mathscr{L}u + \mathscr{L}v\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}(cu) &= (ciu)_t - (cu)_{xx} + \frac{cu}{x}\\ &= ciu_t - cu_{xx} + c\frac{u}{x}\\ &= c\mathscr{L}u.\end{aligned}$$

(h) The order is four, due to the term u_{xxxx}; it is nonlinear, as there is no term that does not depend on u and the operator

$$f(u) = u_t + u_{xxxx} + \sqrt{1+u}$$

is not linear;

$$\begin{aligned} f(2u) &= (2u)_t - (2u)_{xxxx} + \sqrt{1+2u} \\ &= 2u_t - 2u_{xxxx} + \sqrt{1+2u} \\ &\neq 2u_t - 2u_{xxxx} + 2\sqrt{1+u} \\ &= 2f(u). \end{aligned}$$

1.1.4 Suppose that u_1 and u_2 are two solutions of the inhomogeneous linear equation

$$\mathscr{L}\, u = g.$$

It follows that

$$\mathscr{L}\, u_1 = g \qquad \text{and} \qquad \mathscr{L}\, u_2 = g.$$

We have

$$\begin{aligned} \mathscr{L}(u_1 - u_2) &= \mathscr{L}\, u_1 + \mathscr{L}(-u_2) \\ &= \mathscr{L}\, u_1 - \mathscr{L}\, u_2 \\ &= g - g \\ &= 0. \end{aligned}$$

Thus $u_1 - u_2$ is a solution of the homogeneous linear equation

$$\mathscr{L}\, u = 0.$$

1.1.12 Suppose that

$$u(x,y) = \sin nx \sinh ny$$

Then

$$u_x = n \cos nx \sinh ny \qquad \text{and} \qquad u_y = n \sin nx \cosh ny.$$

It follows that

$$u_{xx} = -n^2 \sin nx \sinh ny \qquad \text{and} \qquad u_y = n^2 \sin nx \sinh ny.$$

Hence

$$u_{xx} + u_{yy} = 0,$$

so that

$$u(x,y) = \sin nx \sinh ny$$

is a solution of Laplace's equation.
1.2.1 The general solution to the PDE

$$2u_t + 3u_x = 0,$$

is

$$u(x,t) = f(2x - 3t).$$

If we impose the auxiliary condition then we get

$$\begin{aligned} \sin x &= u(x,0) \\ &= f(2x). \end{aligned}$$

If we put

$$w = 2x \qquad \text{so that} \qquad x = \frac{w}{2}$$

then we get

$$f(w) = \sin \frac{w}{2}.$$

Thus

$$u(x,t) = \sin\left(\frac{2x-3t}{2}\right).$$

1.2.2 Let $v = u_y$. Then

$$\begin{aligned} 0 &= 3u_y + u_{xy} \\ &= 3v + v_x. \end{aligned}$$

We have a linear equation for v,

$$v_x + 3v = 0.$$

Solving this like we would an ODE we get the general solution

$$v(x,y) = f(y)e^{-3x}.$$

This gives us a PDE for u,

$$u_y = f(y)e^{-3x}.$$

This has general solution

$$u(x,y) = F(y)e^{-3x} + G(x),$$

where F and G are arbitrary functions of one variable.
1.2.6 At the point (x, y) the characteristic curve has tangent vector

$$(\sqrt{1-x^2}, 1).$$

Thus the characteristic curve is a solution of the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}.$$

The characteristic curves therefore have equation

$$y = \arcsin x + c$$

so that the general solution is

$$u(x,y) = f(y - \arcsin y)$$

where f is an arbitrary function of one variable.

If we impose the auxiliary condition then we get

$$\begin{aligned} y &= u(0, y) \\ &= f(y). \end{aligned}$$

Thus the solution is

$$u(x, y) = y - \arcsin(x).$$

1.2.9 The PDE

$$u_x + u_y = 1$$

is inhomogeneous linear. A particular solution of this PDE is $u(x, y) = x$, as then

$$u_x = 1 \qquad \text{and} \qquad u_y = 0.$$

The associated homogeneous linear equation

$$u_x + u_y = 0$$

has general solution

$$u(x, y) = f(x - y),$$

where f is an arbitrary function of one variable. It follows that the general solution of the inhomogeneous linear equation is

$$u(x, y) = x + f(x - y),$$

where f is an arbitrary function of one variable.
1.3.6 The three dimensional heat equation is

$$c\rho u_t = \nabla \cdot (\kappa \nabla u)$$

We assume that κ is constant, so that the PDE reduces to

$$u_t = k\Delta u.$$

If we make the axis of the cylinder the z-axis then cylindrical coordinates use the coordinates r, θ and z, where r and θ are polar coordinates for x and y:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \text{and} \qquad z = z.$$

By asumption $u_{zz} = 0$ and $u_\theta = 0$. It follows that

$$\Delta u = u_{xx} + u_{yy}.$$

We have

$$r = \sqrt{x^2 + y^2}.$$

The chain rule gives

$$\begin{aligned} u_{xx} &= (u_x)_x \\ &= (xr^{-1}u_r)_x \\ &= r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-2}u_{rr}. \end{aligned}$$

Similarly

$$\begin{aligned} u_{yy} &= (u_y)_y \\ &= (yr^{-1}u_r)_y \\ &= r^{-1}u_r - y^2r^{-3}u_r + y^2r^{-2}u_{rr}. \end{aligned}$$

It follows that

$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} \\ &= r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-3}u_{rr} + r^{-1}u_r - y^2r^{-2}u_r + y^2r^{-2}u_{rr} \\ &= 2r^{-1}u_r - (x^2+y^2)r^{-3}u_r + (x^2+y^2)r^{-2}u_{rr} \\ &= u_{rr} + r^{-1}u_r. \end{aligned}$$

Thus the heat equation reduces to

$$u_t = k(u_{rr} + u_r/r).$$

1.3.7 As before, we assume that κ is constant, so that the PDE for the heat equation reduces to

$$u_t = k\Delta u.$$

In spherical coordinates we have (ρ, θ, ϕ) where ρ is the distance to the origin, θ is the same angle as in cylindrical coordinates and ϕ is the angle from the z-axis. We have

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

By asumption $u_\theta = 0$ and $u_\phi = 0$. We rename $\rho = r$. The chain rule gives

$$\begin{aligned} u_{xx} &= (u_x)_x \\ &= (xr^{-1}u_r)_x \\ &= r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-2}u_{rr}. \end{aligned}$$

Similarly

$$u_{yy} = r^{-1}u_r - y^2r^{-3}u_r + y^2r^{-2}u_{rr} \quad \text{and} \quad u_{zz} = r^{-1}u_r - z^2r^{-3}u_r + z^2r^{-2}u_{rr}.$$

Therefore

$$\begin{aligned}\Delta u &= u_{xx} + u_{yy} + u_{zz} \\ &= r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-3}u_{rr} + r^{-1}u_r - y^2r^{-2}u_r + y^2r^{-2}u_{rr} + r^{-1}u_r - z^2r^{-3}u_r + z^2r^{-2}u_{rr} \\ &= 3r^{-1}u_r - (x^2+y^2+z^2)r^{-3}u_r + (x^2+y^2+z^2)r^{-2}u_{rr} \\ &= u_{rr} + 2r^{-1}u_r.\end{aligned}$$

Thus the heat equation reduces to

$$u_t = k(u_{rr} + 2u_r/r).$$

Challenge Problems: (Just for fun)

1.2.13 We want to solve

$$u_x + 2u_y + (2x-y)u = 2x^2 + 3xy - 2y^2.$$

We use the change of coordinates

$$x' = x + 2y \qquad \text{and} \qquad y' = 2x - y.$$

We have already seen in the lecture notes that

$$u_x + 2u_y = 5u_{x'}.$$

As

$$(x+2y)(2x-y) = 2x^2 - 3xy + y^2.$$

the PDE reduces to

$$5u_{x'} + y'u = x'y'.$$

This is a linear inhomogeneous equation.
We first guess a solution. We try $u(x',y') = x'$. This is not quite right.
If we subtract $5/y'$ we get an exact solution

$$u(x',y') = x' - 5/y'.$$

The associated homogeneous linear equation is

$$5u_{x'} + y'u = 0.$$

Treating this like an ODE, and using separation of variables, the general solution of the homogeneous linear equation is

$$u(x',y') = f(y')e^{-x'y'/5}$$

where f is an arbitrary function of one variable.
Thus the general solution to the inhomogeneous is

$$u(x',y') = f(y')e^{-x'y'/5} + x' - 5/y'$$

Substituting for x and y we get

$$u(x,y) = f(2x-y)e^{-(x+2y)(2x-y)/5} + x + 2y - 1/(2x-y),$$

is the general solution to the original PDE.
1.3.11 Recall the statement of Stokes' theorem. Let C be any closed curve and let S be any surface bounding C. Let $\vec{F}$ be a vector field on S.

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, \mathrm{d}S.$$

By assumption,

$$\nabla \times \vec{v} = 0,$$

so that the RHS is zero. Therefore

$$\oint_C \vec{v} \cdot \mathrm{d}\vec{r} = 0,$$

for any closed curve C.
This means that we can define a scalar function $\phi(x, y, z)$ as follows. Pick a point p of space and pick a curve γ connecting the origin to this point, for example the straight line connecting the origin to this point. Define

$$\phi(x, y, z) = \int_\gamma \vec{v} \cdot \mathrm{d}\vec{r}.$$

If we want to compute the derivative of ϕ in the $\hat{\imath}$ direction, then consider a line starting at $p = (x, y, z)$ parallel to $\hat{\imath}$.

$$\gamma_1(t) = (x + t, y, z,)$$

As the integral around any closed curve is zero, we have

$$\phi(x + t, y, z) - \phi(x, y, z) = \int_{\gamma_1} \vec{v} \cdot \mathrm{d}\vec{r}.$$

Computing the line integral on the RHS the usual way, we get

$$\phi_x = v_1,$$

the first component of $\vec{v}$.
By symmetry we have

$$\nabla \phi = \vec{v}.$$