MODEL ANSWERS TO THE FIRST HOMEWORK

1.1.2 (a) linear;
\[ \mathcal{L}(u + v) = (u + v)_x + x(u + v)_y \]
\[ = u_x + v_x + xu_y + xv_y \]
\[ = \mathcal{L} u + \mathcal{L} v \]

and

\[ \mathcal{L}(cu) = (cu)_x + x(cu)_x \]
\[ = cu_x + cxu_x \]
\[ = c \mathcal{L} u. \]

(b) Not linear;
\[ \mathcal{L}(2u) = (2u)_x + (2u)(2u)_y \]
\[ = 2u_x + 4uu_y \]
\[ \neq 2u_x + 2uu_y \]
\[ = 2 \mathcal{L} u. \]

(d) Not linear;
\[ \mathcal{L}(2u) = (2u)_x + (2u)_y + 1 \]
\[ \neq 2u_x + 2u_y + 2 \]
\[ = 2 \mathcal{L} u. \]

1.1.3 (a) The order is two, due to the term \( u_{xx} \); it is linear inhomogeneous, as we can put the equation into the form
\[ u_t - u_{xx} = -1 \]

and the operator
\[ \mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \]

is linear;
\[ \mathcal{L}(u + v) = (u + v)_t - (u + v)_{xx} \]
\[ = u_t + v_t - u_{xx} - v_{xx} \]
\[ = \mathcal{L} u + \mathcal{L} v \]
and
\[ \mathcal{L}(cu) = (cu)_t - (cu)_{xx} \]
\[ = cu_t - cu_{xx} \]
\[ = c \mathcal{L} u. \]

(c) The order is three, due to the term \( u_{xxx} \); it is nonlinear, as there is no term that does not depend on \( u \) and the operator
\[ f(u) = u_t - u_{xx} + uu_x \]
is not linear;
\[ f(2u) = (2u)_t - (2u)_{xx} + (2u)(2u)_x \]
\[ = 2u_t - 2u_{xx} + 4uu_x \]
\[ \neq 2u_t - 2u_{xx} + 2uu_x \]
\[ = 2f(u). \]

(e) The order is two, due to the term \( u_{xx} \); it is linear homogeneous, as the operator
\[ \mathcal{L} = i \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{u}{x} \]
is linear;
\[ \mathcal{L}(u + v) = i(u + v)_t - (u + v)_{xx} + \frac{(u + v)}{x} \]
\[ = iu_t + iv_t - v_{xx} - u_{xx} + \frac{u}{x} + \frac{v}{x} \]
\[ = \mathcal{L} u + \mathcal{L} v \]
and
\[ \mathcal{L}(cu) = (ciu)_t - (cu)_{xx} + \frac{cu}{x} \]
\[ = ciu_t - cu_{xx} + \frac{u}{x} \]
\[ = c \mathcal{L} u. \]

(h) The order is four, due to the term \( u_{xxxx} \); it is nonlinear, as there is no term that does not depend on \( u \) and the operator
\[ f(u) = u_t + u_{xxxx} + \sqrt{1 + u} \]
is not linear;
\[ f(2u) = (2u)_t - (2u)_{xxxx} + \sqrt{1 + 2u} \]
\[ = 2u_t - 2u_{xxxx} + \sqrt{1 + 2u} \]
\[ \neq 2u_t - 2u_{xxxx} + 2\sqrt{1 + u} \]
\[ = 2f(u). \]

1.1.4 Suppose that \( u_1 \) and \( u_2 \) are two solutions of the inhomogeneous linear equation
\[ \mathcal{L} u = g. \]
It follows that
\[ \mathcal{L} u_1 = g \quad \text{and} \quad \mathcal{L} u_2 = g. \]
We have
\[ \mathcal{L}(u_1 - u_2) = \mathcal{L} u_1 + \mathcal{L}(-u_2) \]
\[ = \mathcal{L} u_1 - \mathcal{L} u_2 \]
\[ = g - g \]
\[ = 0. \]
Thus \( u_1 - u_2 \) is a solution of the homogeneous linear equation
\[ \mathcal{L} u = 0. \]

1.1.12 Suppose that
\[ u(x, y) = \sin nx \sinh ny \]
Then
\[ u_x = n \cos nx \sinh ny \quad \text{and} \quad u_y = n \sin nx \cosh ny. \]
It follows that
\[ u_{xx} = -n^2 \sin nx \sinh ny \quad \text{and} \quad u_y = n^2 \sin nx \sinh ny. \]
Hence
\[ u_{xx} + u_{yy} = 0, \]
so that
\[ u(x, y) = \sin nx \sinh ny \]
is a solution of Laplace’s equation.

1.2.1 The general solution to the PDE
\[ 2u_t + 3u_x = 0, \]
is
\[ u(x, t) = f(2x - 3t). \]
If we impose the auxiliary condition then we get
\[
\sin x = u(x, 0) = f(2x).
\]
If we put \( w = 2x \) so that \( x = \frac{w}{2} \)
then we get
\[
f(w) = \sin \frac{w}{2}.
\]
Thus
\[
u(x, t) = \sin \left( \frac{2x - 3t}{2} \right).
\]
1.2.2 Let \( v = u_y \). Then
\[
0 = 3u_y + u_{xy} = 3v + v_x.
\]
We have a linear equation for \( v \),
\[
v_x + 3v = 0.
\]
Solving this like we would an ODE we get the general solution
\[
v(x, y) = f(y)e^{-3x}.
\]
This gives us a PDE for \( u \),
\[
u_y = f(y)e^{-3x}.
\]
This has general solution
\[
u(x, y) = F(y)e^{-3x} + G(x),
\]
where \( F \) and \( G \) are arbitrary functions of one variable.
1.2.6 At the point \( (x, y) \) the characteristic curve has tangent vector
\[
(\sqrt{1-x^2}, 1).
\]
Thus the characteristic curve is a solution of the ODE
\[
\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.
\]
The characteristic curves therefore have equation
\[
y = \arcsin x + c
\]
so that the general solution is
\[
u(x, y) = f(y - \arcsin y)
\]
where \( f \) is an arbitrary function of one variable.
If we impose the auxiliary condition then we get
\[ y = u(0, y) = f(y). \]

Thus the solution is
\[ u(x, y) = y - \arcsin(x). \]

1.2.9 The PDE
\[ u_x + u_y = 1 \]
is inhomogeneous linear. A particular solution of this PDE is \( u(x, y) = x \), as then
\[ u_x = 1 \quad \text{and} \quad u_y = 0. \]
The associated homogeneous linear equation
\[ u_x + u_y = 0 \]
has general solution
\[ u(x, y) = f(x - y), \]
where \( f \) is an arbitrary function of one variable. It follows that the general solution of the inhomogeneous linear equation is
\[ u(x, y) = x + f(x - y), \]
where \( f \) is an arbitrary function of one variable.

1.3.6 The three dimensional heat equation is
\[ c \rho u_t = \nabla \cdot (\kappa \nabla u) \]
We assume that \( \kappa \) is constant, so that the PDE reduces to
\[ u_t = k \Delta u. \]
If we make the axis of the cylinder the \( z \)-axis then cylindrical coordinates use the coordinates \( r, \theta \) and \( z \), where \( r \) and \( \theta \) are polar coordinates for \( x \) and \( y \):
\[ x = r \cos \theta \quad y = r \sin \theta \quad \text{and} \quad z = z. \]
By assumption \( u_{zz} = 0 \) and \( u_{\theta} = 0 \). It follows that
\[ \Delta u = u_{xx} + u_{yy}. \]
We have
\[ r = \sqrt{\frac{x^2 + y^2}{5}}. \]
The chain rule gives

\[ u_{xx} = (u_x)_x = (x^{-1}u_r)_x = r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-2}u_{rr}. \]

Similarly

\[ u_{yy} = (u_y)_y = (y^{-1}u_r)_y = r^{-1}u_r - y^2r^{-3}u_r + y^2r^{-2}u_{rr}. \]

It follows that

\[ \Delta u = u_{xx} + u_{yy} = r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-3}u_{rr} + r^{-1}u_r - y^2r^{-2}u_r + y^2r^{-2}u_{rr} = 2r^{-1}u_r - (x^2 + y^2)r^{-3}u_r + (x^2 + y^2)r^{-2}u_{rr} = u_{rr} + r^{-1}u_r. \]

Thus the heat equation reduces to

\[ u_t = k(u_{rr} + u_r/r). \]

1.3.7 As before, we assume that \( \kappa \) is constant, so that the PDE for the heat equation reduces to

\[ u_t = k\Delta u. \]

In spherical coordinates we have \((\rho, \theta, \phi)\) where \(\rho\) is the distance to the origin, \(\theta\) is the same angle as in cylindrical coordinates and \(\phi\) is the angle from the z-axis. We have

\[ \rho = \sqrt{x^2 + y^2 + z^2} \]

By assumption \(u_\theta = 0\) and \(u_\phi = 0\). We rename \(\rho = r\). The chain rule gives

\[ u_{xx} = (u_x)_x = (x^{-1}u_r)_x = r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-2}u_{rr}. \]

Similarly

\[ u_{yy} = r^{-1}u_r - y^2r^{-3}u_r + y^2r^{-2}u_{rr} \quad \text{and} \quad u_{zz} = r^{-1}u_r - z^2r^{-3}u_r + z^2r^{-2}u_{rr}. \]
Therefore
\[ \Delta u = u_{xx} + u_{yy} + u_{zz} \]
\[ = r^{-1}u_r - x^2r^{-3}u_r + x^2r^{-3}u_{rr} + r^{-1}u_r - y^2r^{-2}u_r + y^2r^{-2}u_{rr} + r^{-1}u_r - z^2r^{-3}u_r + z^2r^{-2}u_{rr} \]
\[ = 3r^{-1}u_r - (x^2 + y^2 + z^2)r^{-3}u_r + (x^2 + y^2 + z^2)r^{-2}u_{rr} \]
\[ = u_{rr} + 2r^{-1}u_r. \]

Thus the heat equation reduces to
\[ u_t = k(u_{rr} + 2u_r/r). \]

**Challenge Problems:** (Just for fun)

1.2.13 We want to solve
\[ u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2. \]

We use the change of coordinates
\[ x' = x + 2y \quad \text{and} \quad y' = 2x - y. \]

We have already seen in the lecture notes that
\[ u_x + 2u_y = 5u_{y'}. \]

As
\[ (x + 2y)(2x - y) = 2x^2 - 3xy + y^2. \]

the PDE reduces to
\[ 5u_{y'} + y'u = x'y'. \]

This is a linear inhomogeneous equation.

Note that \( u(x', y') = x' \) is a solution to this PDE. The associated homogeneous linear equation is
\[ 5u_{y'} + y'u = 0. \]

Treating this like an ODE, and using separation of variables, the general solution of the homogeneous linear equation is
\[ u(x', y') = f(x')e^{(-y')^2/10} \]
where \( f(x') \) is an arbitrary function of one variable.

Thus the general solution to the inhomogeneous is
\[ u(x', y') = f(x')e^{(-y')^2/10} + x' \]

Substituting for \( x \) and \( y \) we get
\[ u(x, y) = f(x + 2y)e^{-(2x-y)^2/10} + x + 2y, \]
is the general solution to the original PDE.
1.3.11 Recall the statement of Stokes’ theorem. Let \( C \) be any closed curve and let \( S \) be any surface bounding \( C \). Let \( \vec{F} \) be a vector field on \( S \).

\[
\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS.
\]

By assumption,

\[
\nabla \times \vec{v} = 0,
\]

so that the RHS is zero. Therefore

\[
\oint_C \vec{v} \cdot d\vec{r} = 0,
\]

for any closed curve \( C \).

This means that we can define a scalar function \( \phi(x, y, z) \) as follows. Pick a point \( p \) of space and pick a curve \( \gamma \) connecting the origin to this point, for example the straight line connecting the origin to this point. Define

\[
\phi(x, y, z) = \int_\gamma \vec{v} \cdot d\vec{r}.
\]

If we want to compute the derivative of \( \phi \) in the \( \hat{i} \) direction, then consider a line starting at \( p = (x, y, z) \) parallel to \( \hat{i} \).

\[
\gamma_1(t) = (x + t, y, z,)
\]

As the integral around any closed curve is zero, we have

\[
\phi(x + t, y, z) - \phi(x, y, z) = \int_{\gamma_1} \vec{v} \cdot d\vec{r}.
\]

Computing the line integral on the RHS the usual way, we get

\[
\phi_x = v_1,
\]

the first component of \( \vec{v} \).

By symmetry we have

\[
\nabla \phi = \vec{v}.
\]