MODEL ANSWERS TO THE EIGHTH HOMEWORK

4.2.2. (a) We want to solve

$$X'' = -\lambda X,$$

subject to $X'(0) = X(l) = 0$. Suppose that $\lambda = \beta^2 > 0$.
The general solution of the ODE is

$$X(x) = C \cos \beta x + D \sin \beta x.$$

The boundary conditions imply

$$0 = X'(0) = D\beta$$

and

$$0 = X(l) = C \cos \beta l + D \sin \beta l.$$

The first equation implies that $D = 0$ and so the second equation implies that

$$\cos \beta l = 0.$$

But then

$$\beta l = (n + \frac{1}{2})\pi.$$

It follows that

$$\beta = (n + \frac{1}{2})\frac{\pi}{l} \qquad \text{so that} \qquad \lambda = (n + \frac{1}{2})^2 \frac{\pi^2}{l^2}.$$

The corresponding eigenfunction is then

$$\cos(n + \frac{1}{2})\frac{\pi x}{l}.$$

It is easy to see that λ cannot be zero. One can also easily rule out $\lambda < 0$.
(b) The equation for T is

$$T'' = -\lambda T,$$

This has general solution

$$A_n \cos(n + \frac{1}{2})\frac{\pi t}{l} + B_n \sin(n + \frac{1}{2})\frac{\pi t}{l}.$$

Therefore we have

$$u(x,t) = \sum_n (A_n \cos(n + \frac{1}{2})\frac{\pi t}{l} + B_n \sin(n + \frac{1}{2})\frac{\pi t}{l}) \cos(n + \frac{1}{2})\frac{\pi x}{l}.$$

If we plug in $t = 0$ we get

$$\phi(x) = \sum_n A_n \cos(n+\frac{1}{2})\frac{\pi x}{l} \qquad \text{and} \qquad \psi(x) = \sum_n B_n(n+\frac{1}{2})\frac{\pi}{l}\cos(n+\frac{1}{2})\frac{\pi x}{l}.$$

4.2.4. We want to solve

$$X'' = -\lambda X,$$

subject to $X(-l) = X(l)$ and $X'(-l) = X'(l)$. Suppose first that $\lambda = \beta^2 > 0$.

The general solution of the ODE is

$$X(x) = A\cos\beta x + B\sin\beta x.$$

The boundary conditions imply that

$$A\cos\beta l - B\sin\beta l = A\cos\beta l + B\sin\beta l,$$

and

$$-A\beta\sin\beta l + B\beta\cos\beta l = A\beta\sin\beta l + B\beta\cos\beta l.$$

These equations reduce to

$$B\sin\beta l = 0 \qquad \text{and} \qquad A\sin\beta l = 0.$$

As not both A and B are zero we must have

$$\sin\beta l = 0.$$

But then

$$\beta = \frac{n\pi}{l}$$

and

$$\lambda = \left(\frac{n\pi}{l}\right)^2.$$

Now suppose that $\lambda = 0$. The general solution of the ODE is

$$X(x) = Ax + B.$$

The boundary conditions imply that

$$-lA + B = lA + B \qquad \text{and} \qquad A = A.$$

Thus $A = 0$. It follows that $X(x) = 1$ is an eigenfunction with eigenvalue 0.

If $\lambda < 0$ then the general solution of the ODE is

$$X(x) = A\cosh\beta x + B\sinh\beta x.$$

As cosh and sinh are not periodic, the boundary conditions imply that $A = B = 0$. Thus the eigenvalues are given by

$$\lambda = \left(\frac{n\pi}{l}\right)^2$$

$n = 0, 1, 2, \ldots$.

(b) Given n, the solution of the ODE

$$T' = -\lambda T$$

is

$$T_n(t) = e^{-n^2\pi^2 kt/l^2}.$$

Thus the general solution of the diffusion equation with periodic boundary conditions is

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right) e^{-n^2\pi^2 kt/l^2}.$$

4.3.1. We want to solve the ODE

$$X'' = -\lambda X,$$

subject to $X(0) = 0$ and $X'(l) + aX(l) = 0$ ($a \neq 0$). Assume that $\lambda > 0$. The general solution of the ODE is

$$X(x) = A \cos \beta x + B \sin \beta x,$$

where $\lambda = \beta^2 > 0$. The condition $X(0) = 0$ implies that $A = 0$. The condition that $X'(l) + aX(l) = 0$ implies that

$$B\beta \cos \beta l + Ba \sin \beta l = 0.$$

This reduces to

$$\tan \beta l = -\frac{\beta}{a}.$$

The RHS represents a line through the origin. This meets the graph of $\tan \beta l$, where $\beta > 0$, at infinitely many points. Suppose the solutions are $\beta_1, \beta_2, \dots$.

There are two cases. If $a < 0$ then the slope is positive and there is one solution β_1 between π and $3\pi/2$, one solution β_2 between 2π and $5\pi/2$ and so on,

$$\lim_{m \to \infty} (m + 1/2)\pi - \beta_m = 0.$$

If $a > 0$ then the slope is negative and there is one solution β_1 between $\pi/2$ and π, one solution β_2 between $3\pi/2$ and 2π and so on,

$$\lim_{m \to \infty} \beta_m - (m - 1/2)\pi = 0.$$

Now suppose that $\lambda = 0$. Then $X(x) = Ax + B$. The condition $X(0) = 0$ implies that $B = 0$ and then the condition that $X'(l) + aX(l) = 0$ implies that $A = 0$. There are no eigenfunctions with eigenvalue zero. Finally suppose that $\lambda < 0$. The general solution of the ODE is

$$X(x) = A \cosh \beta x + B \sinh \beta x,$$

where $-\lambda = \beta^2 > 0$. The condition that $X(0) = 0$ implies that $A = 0$. The condition that $X'(l) + aX(l) = 0$ implies that

$$B\beta \cosh \beta l + Ba \sinh \beta l = 0.$$

But then

$$\tanh \beta l = -\frac{\beta}{a}.$$

There are two cases. If $a < 0$ there is one solution. If $a > 0$ there are no solutions.

Thus there is only one negative eigenvalue and only if $a < 0$.

4.3.2. (a) Suppose that $\lambda = 0$. The general solution of the ODE

$$X'' = 0$$

is $X(x) = Cx + D$. The boundary conditions imply that

$$C - a_0 D = 0 \qquad \text{and} \qquad C + a_l(Cl + D) = 0.$$

From the first equation we get $C = a_0 D$. The equation then reduces to

$$a_0 D + a_l(a_0 Dl + D) = 0.$$

Cancelling D we get

$$a_0 + a_l + a_0 a_l l = 0.$$

Conversely if $X(x) = a_0 x + 1$ and $a_0 + a_l = -a_0 a_l l$ then $X(x)$ is an eigenfunction with eigenvalue 0.

(b) The eigenfunctions are $X(x) = a_0 x + 1$.

4.3.11. (a) We have

$$\begin{aligned}
\frac{c^{-2}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l u_t^2 \,\mathrm{d}x &= c^{-2}\int_0^l u_t u_{tt}\,\mathrm{d}x \\
&= \int_0^l u_t u_{xx}\,\mathrm{d}x \\
&= \Big[u_t u_x\Big]_0^l - \int_0^l u_{xt} u_x \,\mathrm{d}x \\
&= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l u_x^2\,\mathrm{d}x.
\end{aligned}$$

To get from the third line to the fourth line we use the fact that $u_t = 0$ at the boundary points, as $u = 0$ on the boundary. Thus the derivative of E with respect to t is zero, so that E is constant in time.

(b) The same calculation is still valid. To get from the third line to the fourth line we use the fact that $u_x = 0$ at the boundary points.

(c) Now we have

$$\Big[u_t u_x\Big]_0^l = u_t(l,t)u_x(l,t) - u_t(0,t)u_x(0,t)$$
$$= -a_l u_t(l,t)u(l,t) - a_0 u_t(0,t)u(0,t)$$
$$= -\frac{1}{2}a_l(u^2(l,t))_t - \frac{1}{2}a_0 u_t(0,t)(u^2(0,t))_t.$$

Thus the derivative of E_R with respect to time is zero, so that E_R is constant.

Challenge Problems: (Just for fun)

4.3.12. (a) The general solution

$$v_{xx} = 0$$

is $v(x) = ax + b$. The boundary conditions imply that

$$a = a = \frac{al + b - b}{l}.$$

Thus $v(x) = 1$ and $v(x) = x$ are two eigenfunctions with eigenvalue 0.
(b) If $\lambda = \beta^2 > 0$ then the general solution

$$v_{xx} = -\lambda v$$

is $v(x) = a\cos\beta x + b\sin\beta x$. The boundary conditions imply that

$$b\beta = -a\beta\sin\beta l + b\beta\cos\beta l = \frac{a\cos\beta l + b\sin\beta l - a}{l}.$$

If we use the first equation to solve for a we get

$$a = b\frac{(\cos\beta l - 1)}{\sin\beta l}.$$

If we use the second equation to solve for a we get

$$a = b\frac{l\beta - \sin\beta l}{\cos\beta l - 1}.$$

Since not both a and b are zero, comparing we get

$$\frac{(\cos\beta l - 1)}{\sin\beta l} = \frac{l\beta - \sin\beta l}{\cos\beta l - 1}$$

so that

$$(\cos\beta l - 1)^2 = \sin\beta l(l\beta - \sin\beta l).$$

(c) If we put

$$\gamma = \frac{1}{2}l\beta.$$

then the equation above reduces to

$$(\cos 2\gamma - 1)^2 = \sin 2\gamma(2\gamma - \sin 2\gamma).$$

Using the double angle formulae this gives

$$4\sin^4\gamma = 2\sin\gamma\cos\gamma(2\gamma - 2\sin\gamma\cos\gamma).$$

Cancelling gives

$$\sin^4\gamma = \sin\gamma\cos\gamma(\gamma - \sin\gamma\cos\gamma).$$

Expanding we get

$$\sin^4\gamma = \gamma\cos\gamma\sin\gamma - \sin^2\gamma\cos^2\gamma.$$

Thus

$$\sin^2\gamma = \gamma\cos\gamma\sin\gamma.$$

(d) One possibility is that $\sin\gamma = 0$, so that

$$\gamma = n\pi,$$

is a multiple of π. Otherwise

$$\sin\gamma = \gamma\cos\gamma.$$

As not both sine and cosine can be zero, we have

$$\tan\gamma = \gamma.$$

Looking at the graph of $\tan\gamma$ versus the graph of γ, we see that there are infinitely many positive solutions $\gamma_1, \gamma_2, \dots$ of the equation. We have

$$\pi \leq \gamma_1 \frac{3\pi}{2} \qquad 2\pi \leq \gamma_2 \frac{5\pi}{2}, \dots$$

and the limit

$$\lim_{n\to\infty} \frac{2n+1}{\pi}/2 - \gamma_n = 0.$$

(e) If $\gamma = n\pi$ then the eigenfunctions are

$$\cos\frac{2n\pi x}{l}.$$

Otherwise the eigenfunctions are

$$(\cos\frac{2\gamma_n}{l} - 1)\cos\frac{2\gamma_n x}{l} + \sin\frac{2\gamma_n}{l}\sin\frac{2\gamma_n x}{l}.$$

Finally, if $\lambda = 0$ then we have

$$1 \qquad \text{and} \qquad x.$$

(f) The general solution is

$$u(x,t) = Ax + B + \sum_n A_n e^{-4n^2\pi^2 kt/l^2}\cos\frac{2n\pi x}{l} + B_n e^{-4\gamma_n^2 kt/l^2}\left((\cos\frac{2\gamma_n}{l} - 1)\cos\frac{2\gamma_n x}{l} + \sin\frac{2\gamma_n}{l}\sin$$

If we set $t = 0$ this reduces to

$$\phi(x) = \sum_n A_n \cos\frac{2n\pi x}{l} + B_n\left((\cos\frac{2\gamma_n}{l} - 1)\cos\frac{2\gamma_n x}{l} + \sin\frac{2\gamma_n}{l}\sin\frac{2\gamma_n x}{l}\right),$$

and this determines the coefficients, $A_1, A_2, \dots$ and $B_1, B_2, \dots$.
The limit as $t \to \infty$ is $Ax + B$.
4.3.13. (a) The only issue is to determine the boundary condition at $x = l$. We assume that the mass is sufficiently small in comparison to the tension, so that we can ignore the effect of gravity. Newton's second law implies that

$$T\frac{u_x}{\sqrt{1 + u_x^2}} = mu_{tt}(l, t),$$

where T is the tension. The denominator of the fraction on the LHS is approximately one, so that this reduces to

$$u_{tt}(l, t) = ku_x(l, t)$$

where

$$k = \frac{T}{m}.$$

(b) Suppose we have a separated solution

$$u(x, t) = X(x)T(t).$$

As usual the wave equation reduces to

$$X'' = -\lambda X \qquad \text{and} \qquad T'' = -\lambda T.$$

The boundary conditions become $X(0) = 0$ and

$$T''(t)X(l) = T(t)X'(l).$$

Using the fact that $T''(t) = -\lambda T(t)$ this reduces to

$$X'(l) = -\lambda X(l).$$

(c) We have

$$X(x) = C\cos\beta x + D\sin\beta x,$$

where $\lambda = \beta^2 > 0$. The first boundary condition implies that

$$C = 0.$$

In this case we may assume that $D = 1$. The second boundary condition then reduces to

$$\beta\cos\beta l = -\beta^2\sin\beta l.$$

As not both sin and cosine can be zero this reduces to

$$\tan\beta l = -\frac{1}{\beta}.$$

It is is not hard to see there are infinitely many solutions $\beta_1, \beta_2, \ldots$. The corresponding eigenfunctions are then

$$\sin \beta_n x.$$