1. Show the following functions are harmonic and find harmonic conjugates:
   (a) \( x^2 - y^2 \)
   (b) \( xy + 3x^2y - y^3 \)
   (c) \( \sinh x \sin y \)
   (d) \( \frac{x}{x^2 + y^2} \)

2. Show that if \( v \) is harmonic conjugate for \( u \) then \(-u\) is a harmonic conjugate of \( v \).

3. (a) Show that Laplace’s equation in polar coordinates is
   \[
   \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} + \frac{\partial^2 u}{\partial \theta^2} = 0.
   \]

   (b) Show that \( \log |z| \) is harmonic on the punctured plane \( \mathbb{C} \setminus \{0\} \).
   (c) Show that \( \log |z| \) has no conjugate harmonic function on the punctured plane \( \mathbb{C} \setminus \{0\} \), but that it does on the plane minus the non-positive reals, \( \mathbb{C} \setminus (-\infty, 0] \).
   (d) Show that \( u(re^{i\theta}) = \theta \log r \) is harmonic. Find a harmonic conjugate \( v \) for \( u \), using the polar form of the Cauchy-Riemann equations. What is the holomorphic function \( u + iv \)?

4. Let \( u \) be a harmonic function on the annulus
   \[
   \{ z \in \mathbb{C} \mid a < |z| < b \}.
   \]
   Show that there is a constant \( C \) such that \( u(z) - C \log |z| \) has a harmonic conjugate on the annulus. Show that this constant is
   \[
   C = \frac{r}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial r}(re^{i\theta}) \, d\theta,
   \]
   where \( r \in (a, b) \).

5. Let \( U \) be a bounded region and let \( u \) be a harmonic function that extends continuously to the boundary \( \partial U \) of \( U \).
   Show that if \( u \in [a, b] \) on \( \partial U \) then \( u \in [a, b] \) on the whole of \( U \).
6. Fix \( n \geq 1, r > 0 \) and \( \lambda = \rho e^{i\theta} \). What is the maximum modulus of \( z^n + \lambda \) over the closed disk 
\[
\{ z \in \mathbb{C} \mid |z| \leq r \}.
\]
Where does \( z^n + \lambda \) attain its maximum over this disk?

7. Let \( f(z) \) be a holomorphic function on a region \( U \) that is nowhere zero on \( U \).
   (a) Show that if \( |f(z)| \) attains its minimum on \( U \) then \( f(z) \) is constant.
   (b) If \( U \) is bounded and \( f(z) \) extends to a continuous function on the boundary of \( U \) then \( |f(z)| \) attains its minimum on \( \partial U \).

8. Let \( f(z) \) be a bounded holomorphic function on the right half plane. Suppose that \( f(z) \) extends continuously to the imaginary axis and that 
   \[
   |f(iy)| \leq M
   \]
   for all points \( iy \) on the imaginary axis. Show that 
   \[
   |f(z)| \leq M
   \]
   for all \( z \) in the right half plane. \((\text{Hint: consider } (z + 1)^{-\epsilon} f(z) \text{ on a large half disk, where } \epsilon > 0 \text{ is small}).\)

**Challenge Problems:** (Just for fun)

9. Prove the fundamental theorem of algebra by applying the maximum principle to \( 1/p(z) \) on a disk of large radius.