HOMEWORK 6, DUE WEDNESDAY MAY 13TH, 12PM

1. Suppose that f is a holomorphic function on the open disk

$$U = \{ z \in \mathbb{C} \,|\, |z - a| < R \}$$

of radius R centred about a such that

$$|f(z)| \leq M$$

on U.
Show that if $f(z)$ has a zero of order m at a then

$$|f(z)| \leq \frac{M}{R^m}|z - a|^m$$

on U, with equality at $z \neq a$ if and only if $f(z)$ is a scalar multiple of $(z - a)^m$.
2. Suppose that $f(z)$ is a holomorphic function of the unit disk Δ to itself. Show that if $f(z)$ has a zero of order m at a then

$$|f(z)| \leq |\psi(z)|^m \qquad \text{where} \qquad \psi(z) = \frac{z - a}{1 - \bar{a}z}.$$

Conclude that

$$|f(0)| \leq |a|^m.$$

3. Suppose that $f(z)$ is a holomorphic function on the closed unit disk and that

$$1 < |f(z)| < M \qquad \text{where} \qquad |z| = 1.$$

If $f(0) = 1$ then show that $f(z)$ has a zero in the unit disk and that all such zeroes a have the property that

$$|a| > \frac{1}{M}.$$

4. A **finite Blaschke product** is a rational function of the form

$$B(z) = e^{i\varphi}\left(\frac{z - a_1}{1 - \bar{a}_1 z}\right)\left(\frac{z - a_2}{1 - \bar{a}_2 z}\right)\cdots\left(\frac{z - a_n}{1 - \bar{a}_n z}\right)$$

where $a_1, a_2, \dots, a_n \in \Delta$ and $\varphi \in [0, 2\pi)$.
Show that $f(z)$ is a finite Blaschke product if and only if $f(z)$ is holomorphic on Δ, continuous on the closed unit disk and $|f(z)| = 1$ on the circle $|z| = 1$.
5. Show that

$$f(z) = \frac{1 + 3z^2}{3 + z^2}$$

is a finite Blaschke product.

6. Suppose that f is holomorphic for $|z| < 3$. If $|f(z)| \leq 1$ and $f(\pm i) = f(\pm 1) = 0$, then what is the maximum value of $|f(0)|$? For which functions is the maximum attained?

7. If z_0 and $z_1 \in \Delta$, find the maximum value of

$$|f(z_1) - f(z_0)|$$

as f ranges over all holomorphic functions on Δ such that $|f(z)| < 1$. Determine for which functions the maximum value is attained.

Hint: First consider the case where $z_0 = r$ and $z_1 = -r$. Show that the maximum is $2r$, attained only for $f(z) = \lambda z$, $|\lambda| = 1$.

8. (a) Write down a Möbius transformation that is a biholomorphic map from the unit disk Δ to the upper half plane $\mathbb{H}$.

(b) Show that any biholomorphic map of the upper half plane $\mathbb{H}$ to itself has the form

$$z \longrightarrow \frac{az+b}{cz+d},$$

where a, b, c and d are real numbers such that $ad - bc = 1$.

(c) Show that every biholomorphic map of the upper half plane $\mathbb{H}$ to the unit disk Δ has the form

$$z \longrightarrow e^{i\varphi}\frac{z-a}{z-\bar{a}} \qquad \text{where} \qquad \operatorname{Im} a > 0, \varphi \in [0, 2\pi).$$

(d) Show that a and φ are uniquely determined by the biholomorphic map in (c).

Challenge Problems: (Just for fun)