HOMEWORK 8, DUE WEDNESDAY MAY 27TH, 12PM

1. Check that

(a)

$$\frac{\partial}{\partial\psi}\left(2\arctan\left(\frac{1+r}{1-r}\tan\frac{\psi}{2}\right)\right) = P_r(\psi).$$

(b)

$$\arctan\left(\frac{1+r}{1-r}\tan\frac{2\pi-\theta}{2}\right) - \arctan\left(\frac{1+r}{1-r}\tan\frac{\pi-\theta}{2}\right) = \arctan\left(\frac{1-r^2}{2r\sin\theta}\right).$$

2. Suppose that $f(z) = u(z) + iv(z)$ is a holomorphic function on the unit disk and that $u(z)$ extends to a continuous function on the closed unit disk. Show that

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})\frac{e^{i\phi}+z}{e^{i\phi}-z}\,\mathrm{d}\phi + iv(0).$$

This is known as **Schwarz's formula**. It expresses a holomorphic function as an integral using only its real part on the boundary.

3. Show that

$$u(r,\theta) = \frac{1}{\pi}\arctan\left(\frac{1-x^2-y^2}{(x-1)^2+(y-1)^2-1}\right) \qquad \text{where} \qquad \arctan t \in [0,\pi]$$

is the solution to Dirichlet's problem for the unit disk for the piecewise continuous function

$$h(e^{i\phi}) = \begin{cases} 1 & \text{if } \phi \in (0,\pi/2) \\ 0 & \text{if } \phi \in (\pi/2, 2\pi). \end{cases}$$

(a) Using the Poisson kernel.

(b) Using a biholomorphic map to change the region to the upper half plane.

4. Show that

$$u(r,\theta) = \frac{1}{\pi}\arctan\left(\frac{(1-x^2-y^2)y_0}{(x-1)^2+(y-y_0)^2-y_0^2}\right) \qquad \text{where} \qquad \arctan t \in [0,\pi]$$

is the solution to Dirichlet's problem for the unit disk for the piecewise continuous function

$$h(e^{i\phi}) = \begin{cases} 1 & \text{if } \phi \in (0,2\theta_0) \\ 0 & \text{if } \phi \in (2\theta_0, 2\pi) \end{cases}$$

where $\theta_0 \in (0, \pi/2)$ and $y_0 = \tan\theta_0$. Check that $u(r,\theta)$ has the correct behaviour at the boundary.
5. Let $\delta_h \colon \mathbb{R} \longrightarrow \mathbb{R}$ be the finite bump function

$$\delta_h(x) = \begin{cases} \frac{1}{h} & \text{if } x \in [0,h] \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\int_{-\infty}^{\infty} \delta_h(x)\,\mathrm{d}x = 1.$$

Show that

$$\int_0^{2\pi} P_r(\phi-\theta)\delta_h(\phi-\theta_0)\,\mathrm{d}\phi = P_r(c-\theta),$$

where $c \in (\theta_0, \theta_0 + h)$. Conclude that

$$\lim_{h\to 0^+} \int_0^{2\pi} P_r(\phi-\theta)\delta_h(\phi-\theta_0)\,\mathrm{d}\phi = P_r(\theta-\theta_0).$$

Hence the Poisson kernel $P_r(\theta-\theta_0)$ is the limit, as h approaches zero from above, of the harmonic function on the unit disk whose boundary values are $2\pi\delta_h(\theta-\theta_0)$.
6. (a) Show that

$$1 + 2\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{1-a^2}{1-2a\cos\theta + a^b} \qquad \text{where} \qquad a \in (-1,1).$$

(*Hint: realise the LHS as the real part of a geometric series.*)
(b) Conclude that

$$P_r(\psi) = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\psi.$$

(c) Show that the solution to Dirichlet's problem on the unit disk is given by

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n \sin n\theta)$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} h(e^{i\phi})\cos n\phi\,\mathrm{d}\phi$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} h(e^{i\phi})\sin n\phi\,\mathrm{d}\phi.$$

Challenge Problems: (Just for fun)