HOMEWORK 9, DUE WEDNESDAY JUNE 3RD, 12PM

1. Show that reflection in the circle

$$\{ z \in \mathbb{C} \,|\, |z - a| = R \}$$

is given by

$$z^* = a + R^2 \frac{z - a}{|z - a|^2}.$$

2. Show that reflection in a circle maps lines and circles in the plane to lines and circles.

3. Let f be an entire function whose modulus is constant on a circle centred at a. Show that

$$f(z) = c(z - a)^n,$$

for some integer $n \geq 0$ and a constant $c \in \mathbb{C}$.

4. Show that if $f(z)$ is a meromorphic function on the unit disk such that

$$\lim_{|z| \to 1} |f(z)| = 1$$

then $f(z)$ is a rational function. Show further that $f(z)$ is the quotient of two finite Blaschke products.

5. The **modulus of an annulus**

$$U = \{ z \in \mathbb{C} \,|\, a < |z - \alpha| < b \}$$

is defined to be

$$\frac{1}{2\pi} \ln \left(\frac{b}{a} \right).$$

(a) Show that any biholomorphic, that is, conformal map from one annulus centred at the origin to another annulus centred at the origin extends to a biholomorphic map of the punctured plane to itself.

(b) Show that two annuli are biholomorphic (or conformal) if and only if the annuli have the same moduli. You may use the fact that any biholomorphic map of the punctured complex plane is a Möbius transformation that either fixes 0 and ∞ or switches them (see Question 7 of Homework 7).

(c) Show that every biholomorphic map of the annulus

$$U = \{ z \in \mathbb{C} \,|\, a < |z| < b \}$$

to itself is either a rotation $z \longrightarrow e^{i\varphi} z$ or a rotation followed by the inversion

$$z \longrightarrow \frac{ab}{z}.$$

6. Let

$$U = \{\, z \in \Delta \mid \mathrm{Re}(z) > 0 \,\}$$

be the right half of the unit disk and let

$$g \colon U \longrightarrow \Delta$$

be *the* biholomorphic map that fixes the points $\pm i$ and 1.
(a) Show that

$$h \colon U \longrightarrow \Delta \qquad \text{given by} \qquad h(z) = \overline{g(\bar{z})}$$

is another biholomorphic that also fixes the points $\pm i$ and 1.
(b) Conclude that

$$g(\bar{z}) = \overline{g(z)}.$$

(c) Use (b) to conclude that $g(0) = -1$.
(d) Write $g(z)$ as an explicit composition of biholomorphic maps and check that $g(0) = -1$.
7. Show that no two of the regions, the unit disk Δ, the complex plane $\mathbb{C}$ and the extended complex plane $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, are biholomorphic.
8. If

$$f \colon U \longrightarrow \Delta$$

is a biholomorphic map of the region U and the unit disk, find a biholomorphic map of the region

$$U_r = \{\, z \in U \mid |f(z)| < r \,\} \qquad \text{where} \qquad r \in (0, 1)$$

and the unit disk.

Challenge Problems: (Just for fun)

9. Consider reflection $z \longrightarrow z^*$ across the parabola $y = x^2$.
(a) Find an expression for z^*.
(b) Expand z^* in a power series in powers of $\bar{z}$.
(c) Determine the radius of convergence of the power series from (b).
(d) Try to explain graphically why the radius of convergence is finite.
10. The words "AMBULANCE" are written backwards on the front of an ambulance so that when a driver sees an ambulance approaching from behind in their rear view mirror the words appear normally. Why aren't the letters upside down as well?