14. The mean value and maximum property

**Definition 14.1.** Let \( h: U \rightarrow \mathbb{R} \) be a continuous, real-valued function on a region \( U \). Suppose that the open disk of radius \( \rho \) centred around \( a \) is contained in \( U \).

The *average value of* \( h \) on the circle \( |z - a| = r \) is defined to be

\[
A(r) = \frac{1}{2\pi} \int_{0}^{2\pi} h(a + re^{i\theta}) \, d\theta \quad \text{where} \quad r \in (0, \rho).
\]

As \( h \) is a continuous function it follows that the average value \( A(r) \) is a continuous function of \( r \). The average of \( h \) at \( a \) is presumably \( h(a) \) and so one might guess that \( A(r) \) approaches \( h(a) \) as \( r \) approaches zero. In fact this is the case:

**Lemma 14.2.** Let \( h: U \rightarrow \mathbb{R} \) be a continuous, real-valued function on a region \( U \).

Then

\[
\lim_{r \to 0} A(r) = h(a) \quad \text{where} \quad A(r) = \frac{1}{2\pi} \int_{0}^{2\pi} h(a + re^{i\theta}) \, d\theta
\]

is the average value.

**Proof.** We have

\[
|A(r) - h(a)| = \left| \frac{1}{2\pi} \int_{0}^{2\pi} h(a + re^{i\theta}) - h(a) \, d\theta \right| = \frac{1}{2\pi} \int_{0}^{2\pi} |h(a + re^{i\theta}) - h(a)| \, d\theta \leq M,
\]

where \( M = M(r) \) is the maximum value of \( |h(a + re^{i\theta}) - h(a)| \). As \( h \) is continuous, \( M \) approaches zero as \( r \) approaches 0. \( \Box \)

**Theorem 14.3.** If \( u \) is a harmonic function on a region \( U \) and if the open disk of radius \( \rho \) centred at \( a \) is contained in \( U \) then

\[
u(a) = \frac{1}{2\pi} \int_{0}^{2\pi} u(a + re^{i\theta}) \, d\theta \quad \text{where} \quad r \in (0, \rho).
\]

**Proof.** We give two proofs.
For the first proof, note that
\[
\oint_{|\mathbf{z}-\mathbf{a}|=r} \frac{\partial u}{\partial y} \, dx + \frac{\partial u}{\partial x} \, dy = \iint_{|\mathbf{z}-\mathbf{a}|<r} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \, dxdy
\]
\[
= \iint_{|\mathbf{z}-\mathbf{a}|<r} 0 \, dxdy
\]
\[
= 0.
\]
The first line follows from Green’s theorem. We parametrise the circle by
\[
x(\theta) = x_0 + r \cos \theta \quad \text{and} \quad y(\theta) = y_0 + r \sin \theta.
\]
It follows that
\[
0 = \oint_{|\mathbf{z}-\mathbf{a}|=r} \frac{\partial u}{\partial y} \, dx + \frac{\partial u}{\partial x} \, dy
\]
\[
= r \int_0^{2\pi} \left[ \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta \right] \, d\theta
\]
\[
= r \int_0^{2\pi} \frac{\partial u}{\partial r} (a + re^{i\theta}) \, d\theta.
\]
Dividing by $2\pi r$ and taking the derivative out of the integral sign gives
\[
\frac{\partial}{\partial r} \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) \, d\theta = 0.
\]
It follows that $A(r)$ is constant. As it approaches $u(a)$ as $r$ goes to zero, we must have
\[
u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) \, d\theta.
\]
For the second proof, pick a harmonic conjugate $v$ and let $f = u + iv$. Then $f$ is holomorphic. Cauchy’s integral formula implies that
\[
f(a) = \frac{1}{2\pi i} \oint_{|\mathbf{z}-\mathbf{a}|=r} \frac{f(z)}{z-a} \, dz.
\]
Suppose that we parametrise the circle
\[
z = a + re^{i\theta} \quad \text{so that} \quad \frac{dz}{i(z-a)} = d\theta.
\]
It follows that
\[
f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) \, d\theta.
\]
Now take real parts.
The property that the average value of \( u \) around a circle is equal to the value at the centre is called the \textit{mean value property}. We have just established that harmonic functions satisfy the mean value property. It is an amazing fact that any continuous function which satisfies the mean value property is in fact harmonic.

**Theorem 14.4** (Strict maximum principle). Let \( u \) be a harmonic function on a region \( U \) such that \( u(z) \leq M \) on \( U \).

If there is a point \( a \in U \) such that \( u(a) = M \) then \( u \) is constant.

\textit{Proof.} Let \( V \subset U \) be the set of points \( b \in U \) such that \( u(b) = M \). By assumption \( V \) is non-empty as \( a \in V \).

Suppose that \( b \in V \). As \( U \) is open we can find an open disk centred at \( b \) contained in \( U \). We have

\[
    u(b) = \frac{1}{2\pi} \int_{0}^{2\pi} u(b + re^{i\theta}) \, d\theta,
\]

for any sufficiently small \( r \). But then

\[
    0 = \frac{1}{2\pi} \int_{0}^{2\pi} u(b) - u(b + re^{i\theta}) \, d\theta,
\]

The integrand

\[
    u(b) - u(b + re^{i\theta}) \geq 0,
\]

by assumption. As the area under the graph is zero the only possibility is that

\[
    u(b) - u(b + re^{i\theta}) = 0
\]

all the way around the boundary.

It follows that every point \( c \) whose distance to \( b \) is \( r \) belongs to \( V \). Varying \( r \) we see that an open disk centred around \( b \) is contained in \( V \). It follows that \( V \) is open and so \( V = U \) as \( U \) is connected.

We can extend the maximum principle for harmonic functions to holomorphic functions

**Theorem 14.5** (Strict maximum principle). Let \( f \) be a holomorphic function on a region \( U \) such that \( |f(z)| \leq M \) on \( U \).

If there is a point \( a \in U \) such that \( |f(a)| = M \) then \( u \) is constant.

\textit{Proof.} Suppose that

\[
    f(a) = Me^{i\theta}.
\]

Let

\[
    g : U \longrightarrow \mathbb{C}
\]

be the function

\[
    z \longrightarrow e^{-i\theta} f(z)
\]
Then
\[ |g(z)| = |f(z)| \quad \text{and} \quad g(a) = M. \]
It follows that \( |g(z)| \leq M \) on \( U \).

Let \( u \) be the real part of \( g = u + iv \). Then \( u \) is a harmonic function on \( U \),

\[
    u(z) \leq |g(z)| \\
    \leq M
\]
and

\[
    u(a) = M.
\]
Thus \( u(z) \) is constant by (14.4). Thus \( u(z) = M \). As \( |g(z)| \leq M \) it follows that \( v = 0 \) and \( g(z) = M \) is constant. But then \( f(z) \) is constant.

\[ \square \]

**Theorem 14.6 (Maximum principle).** Let \( f(z) \) be a holomorphic function on a bounded region \( U \) which extends to a continuous function on the boundary.

If \( |f(z)| \leq M \) on the boundary \( \partial U \) then \( |f(z)| \leq M \) on the whole of \( U \).

**Proof.** The follows from a basic fact in topology that the continuous function \( |f(z)| \) must attain its maximum at some point of \( U \cup \partial U \). If it is on \( U \) then the strong maximum principle implies that \( f \) is constant and the result is clear.

If \( |f(z)| \) attains its maximum on the boundary of \( U \) then the result is also clear. \[ \square \]