18. Meromorphic versus Rational functions

One obvious way to give a meromorphic function on \( \mathbb{C} \) is to write down a rational function, the quotient of two polynomials. It is natural to wonder to what extent these give all meromorphic functions on \( \mathbb{C} \):

**Theorem 18.1.** Let \( f \) be a meromorphic function on the whole of \( \mathbb{C} \). Then \( f \) is a rational function if and only if it has at worse a pole at infinity.

Here we allow the possibility that \( f \) is holomorphic at infinity.

It is clear that we need the condition at infinity. For example the exponential function

\[ z \rightarrow e^z \]

is an entire function but it is not a rational function. In fact it has an essential singularity at infinity.

We will need:

**Definition 18.2.** Let \( f \) be a holomorphic function with an isolated singularity at \( a \).

The principal part of \( f \) at \( a \) is the negative part of the Laurent series expansion of \( f \) at \( a \).

**Proof of (18.1).** One direction is clear. If

\[ f(z) = \frac{p(z)}{q(z)} \]

is a rational function then

\[ f \left( \frac{1}{z} \right) = \frac{p(1/z)}{q(1/z)}. \]

The RHS expands to a rational function in \( z \). In particular \( f(z) \) has a pole at infinity.

Now suppose that \( f(z) \) is a meromorphic function with a pole at infinity. First observe that \( f(z) \) has only finitely many singularities. Isolated singularities cannot accumulate anywhere. If there were infinitely many singularities their modulus would have to go to infinity. But as we have a pole at infinity this cannot happen either.

Let \( w = 1/z \). By assumption

\[ f(w) = \frac{a_{-n}}{w^n} + \frac{a_{-n+1}}{w^{n-1}} + \cdots + a_0 + a_1 w + \ldots. \]

The principal part at \( w = 0 \) is then

\[ \frac{a_{-n}}{w^n} + \frac{a_{-n+1}}{w^{n-1}} + \cdots + \frac{a_{-2}}{w^2} + \frac{a_{-1}}{w}. \]
If we substitute back in \( z = 1/w \) then we get a polynomial in \( z \), \( p_\infty(z) \).

Let \( a_1, a_2, \ldots, a_m \) be the finitely many other singular points. Let \( p_{a_i}(z) \) be the principal part at \( a_i \). Then \( p_{a_i}(z) \) is a polynomial in \( 1/(z-a_i) \).

Consider

\[
g(z) = f(z) - p_\infty(z) - p_{a_1}(z) - p_{a_2}(z) - \cdots - p_{a_m}(z).
\]

Note that

\[
p_\infty(z) + p_{a_1}(z) + p_{a_2}(z) + \cdots + p_{a_m}(z).
\]

is a rational function in \( z \), since it is a sum of rational functions. In particular it is a meromorphic function so that \( g(z) \) is a meromorphic function.

The only possible poles of \( g(z) \) are at \( a_1, a_2, \ldots, a_m \). But by construction the principal part of \( g(z) \) at \( a_i \) is zero. Therefore \( g(z) \) is holomorphic at \( a_i \), so that it is an entire function.

On the other hand, \( g(z) \) does not have a pole at infinity either. Therefore \( g(z) \) is bounded as \( z \) approaches infinity. But then Liouville’s theorem implies that \( g(z) \) is a constant \( c \). It follows that

\[
f(z) = p_\infty(z) + p_{a_1}(z) + p_{a_2}(z) + \cdots + p_{a_m}(z) + c
\]

is a rational function. \( \square \)

It seems worth pointing out the decomposition above in \( p_\infty \) and \( p_{a_1}, p_{a_2}, \ldots, \) is essentially the decomposition of a rational function into partial fractions.

**Example 18.3.** Obtain the partial fraction decomposition of

\[
\frac{z^3}{z^2 + 1}.
\]

The first thing to do is figure out the term \( p_\infty(z) \). We could substitute \( w = 1/z \), find the Laurent series expansion and then isolate the principal part. Much easier is to use the division algorithm. We try to divide \( z^2 + 1 \) into \( z^3 \). We get \( z \) with a remainder of

\[
z^3 - z(z^2 + 1) = -z \quad \text{so that} \quad \frac{z^3}{z^2 + 1} = z - \frac{z}{z^2 + 1}.
\]

Thus \( p_\infty(z) = z \). Now

\[
-\frac{z}{z^2 + 1}
\]
has poles at $\pm i$. We have simple poles there and so

\[ - \frac{z}{z^2 + 1} = p_i(z) + p_{-i}(z) \]

\[ = \frac{\alpha}{z - i} + \frac{\beta}{z + i}. \]

Now $\alpha$ is the residue at $i$ and $\beta$ is the residue at $-i$. We have

\[ \alpha = \lim_{z \to i} -\frac{z}{2z} \]

\[ = -\frac{1}{2}. \]

It is clear the answer won’t change at $-i$. Thus

\[ \frac{z^3}{z^2 + 1} = z - \frac{1}{2} \frac{1}{z - i} - \frac{1}{2} \frac{1}{z + i}. \]

Of course we could do the last step using one of the usual methods.