24. RIEMANN MAPPING THEOREM

**Theorem 24.1.** If $U \subset \mathbb{C}$ is a simply connected region then either $U = \mathbb{C}$ or $U$ is biholomorphic to the unit disk $\Delta$.

Recall that $U$ is simply connected if every closed path can be continuously shrunk to a point. Intuitively this says that $U$ has no holes. Every open disk is simply connected but an annulus is never simply connected. It is theorem that a region is simply connected if and only if the complement in the Riemann sphere is connected.

If we think about trying to construct a biholomorphic map $f : U \to \Delta$ the first thing to realise is that $f$ is not unique. If we compose $f$ with a biholomorphic map $\alpha$ of $\Delta$ to itself we get another biholomorphic map $g = \alpha \circ f$.

Recall that $\alpha$ has the form $z \mapsto e^{i\theta} \frac{z - a}{1 - \bar{a}z}$.

$\alpha$ is determined by two pieces of information, $a$, which is the inverse image of 0 and the argument $\theta$ of the derivative at zero, $e^{i\theta}$.

To make $f$ unique, we pick a point $a$ in $U$ and we require that $f(a) = 0$. Then we require the derivative $f'(a) > 0$ is real and positive. In terms of $\alpha$, this fixes $a$ and $e^{i\theta}$.

**Corollary 24.2.** If $U \subset \mathbb{P}^1$ is a simply connected region in the extended complex plane then either

1. $U = \mathbb{P}^1$, or
2. $U$ is biholomorphic to $\mathbb{C}$, or
3. $U$ is biholomorphic to the unit disk $\Delta$.

**Proof.** Let $Z$ be the complement of $U$. Then $Z$ is empty if and only if $U = \mathbb{P}^1$.

Suppose that $Z$ is not empty. If $a \in Z$ then pick a Möbius transformation that sends $a$ to $\infty$. For example,

$$ z \mapsto \frac{1}{z - a} $$

will do. The image of $U$ under this Möbius transformation is biholomorphic to $U$. Thus we may assume that $\infty \in Z$. But then $U \subset \mathbb{C}$ and the Riemann mapping theorem implies that (2) or (3) holds. $\square$