Example 5.1. Calculate

\[ \int_0^{2\pi} \frac{d\theta}{a + \cos \theta} \quad \text{where} \quad a > 1. \]

We are going to use contour integration to evaluate this integral. Clearly this integral has a different nature to the previous examples, since the range of integration is a finite interval.

However it does seem that we are using angles and that we are going around a circle. So let’s try

\[ U = \Delta, \]

the open unit disk. In this case the boundary is the unit circle. We use the standard parametrisation

\[ z = \gamma(\theta) = e^{i\theta} \quad \text{so that} \quad dz = ie^{i\theta} d\theta = iz \, d\theta. \]

It follows that

\[ d\theta = \frac{dz}{iz}. \]

Now

\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}. \]

This gives

\[ \int_0^{2\pi} \frac{d\theta}{a + \cos \theta} = \int_{|z|=1} \frac{dz}{iz(a + \frac{z + \frac{1}{z}}{2})} = \frac{2}{i} \int_{|z|=1} \frac{dz}{z^2 + 2za + 1} \]

The integrand is

\[ \frac{1}{z^2 + 2za + 1}. \]

This has poles at the zeroes of

\[ z^2 + 2az + 1, \]

which are given by the quadratic formula,

\[ \frac{-2a \pm \sqrt{4a^2 - 4}}{2} = -a \pm \sqrt{a^2 - 1}. \]
Now the negative square root doesn’t belong to \( \Delta \) and the positive square root

\[
\alpha = -a + \sqrt{a^2 - 1}
\]
does belong to \( \Delta \). The positive square root is a simple pole. The residue is

\[
\text{Res}_\alpha f(z) = \lim_{z \to \alpha} \frac{z - \alpha}{z^2 + 2az + 1}
\]

\[
= \lim_{z \to \alpha} \frac{1}{2z + 2a}
\]

\[
= \frac{1}{2(\sqrt{a^2 - 1})}.
\]

The residue theorem implies

\[
\int_0^{2\pi} \frac{d\theta}{a + \cos \theta} = 2\pi i \frac{2}{i} \text{Res}_\alpha f(z)
\]

\[
= \frac{2\pi}{\sqrt{a^2 - 1}}.
\]

**Example 5.2.** Calculate

\[
\int_0^{\pi} \frac{\cos 2\theta d\theta}{1 - 2a \cos \theta + a^2}
\]

where \( a \in (-1, 1) \).

We first suppose that \( a \neq 0 \). Note that the graph of \( \cos \theta \) is symmetric about the vertical line \( \theta = \pi \) so that the integrand is symmetric about the same line.

So we calculate

\[
\int_0^{2\pi} \frac{\cos 2\theta d\theta}{1 - 2a \cos \theta + a^2}
\]

and divide by 2.

We proceed as before, we integrate around the unit circle and we use the same parametrisation. We have

\[
\int_0^{2\pi} \frac{\cos 2\theta d\theta}{1 - 2a \cos \theta + a^2} = \int_\gamma \frac{z^2 + \frac{1}{z^2}}{2iz(1 - a(z + 1/z) + a^2)} dz
\]

\[
= \frac{i}{2} \int_\gamma \frac{z^4 + 1}{z^2(a z^2 + a - z - a^2 z)} dz
\]

\[
= \frac{i}{2} \int_\gamma \frac{z^4 + 1}{z^2(z - a)(az - 1)} dz
\]

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The integrand is
\[ f(z) = \frac{z^4 + 1}{z^2(z - a)(az - 1)}. \]
This has isolated singularities at 0, a and 1/a. Since \( a \in (-1, 1) \), on the first two belong to the unit disk.

The first singularity is a double pole:

\[
\text{Res}_0 f(z) = \lim_{z \to 0} \frac{d}{dz} \left( \frac{z^4 + 1}{(z-a)(az-1)} \right) \\
= \lim_{z \to 0} \frac{4z^3(z-a)(az-1) - (z^4+1)(2az-1-a^2)}{(z-a)^2(az-1)^2} \\
= \frac{a^2 + 1}{a^2}.
\]

The second is a simple pole

\[
\text{Res}_a f(z) = \lim_{z \to a} \frac{z^4 + 1}{z^2(az-1)} \\
= \frac{a^4 + 1}{a^2(a^2 - 1)}.
\]

The residue theorem implies that
\[
\oint_{\gamma} \frac{z^4 + 1}{z^2(z-a)(az-1)} \, dz = 2\pi i \left( \text{Res}_0 f(z) + \text{Res}_a f(z) \right) \\
= 2\pi i \left( \frac{a^2 + 1}{a^2} + \frac{a^4 + 1}{a^2(a^2 - 1)} \right) \\
= 2\pi i \left( \frac{a^4 - 1}{a^2(a^2 - 1)} + \frac{a^4 + 1}{a^2(a^2 - 1)} \right) \\
= 2\pi i \frac{2a^2}{a^2 - 1}.
\]

It follows that
\[
\int_0^\pi \frac{\cos 2\theta \, d\theta}{1 - 2a \cos \theta + a^2} = \frac{i}{4} \cdot 2\pi i \frac{2a^2}{a^2 - 1} \\
= \frac{\pi a^2}{1 - a^2}.
\]