Example 9.1. Calculate

\[ I = \int_{0}^{\infty} \frac{dx}{x^3 + 1}. \]

At first sight this integral seems straightforward, just integrate \( \frac{1}{z^3 + 1} \) over the standard contour. The problem is that if we do this then we get the value of \( \int_{-\infty}^{\infty} \frac{dx}{x^3 + 1} \) and there is no obvious way to go from the value of this integral to the value of the integral we are after, since \( x^3 + 1 \) is neither odd nor even.

We already saw one fix in a homework problem, integrate along an arc instead of a semicircle. If we go along the line from \( e^{2\pi i/3}R \) to 0 then we can exploit the fact that

\[
\frac{1}{(e^{2\pi i/3}t)^3 + 1} = \frac{1}{t^3 + 1}.
\]

Here is another way to proceed which is more versatile. It looks as though using a keyhole contour in Lecture 8 might work, since we only integrate along the interval \([\rho, R]\). The problem with using the keyhole contour is that there is no ambiguity in the definition of \( \frac{1}{z^3 + 1} \) so that when we integrate over \( \gamma_- + \gamma_+ \) the integrals cancel.

To engineer an integral that does not cancel we integrate \( f(z) = \frac{\log z}{z^3 + 1} \) instead. We use the same branch of the logarithm as in Lecture 8. We cut along the positive real axis:

\[
\log z = \ln |z| + i \arg z \quad \text{where} \quad \arg z \in (0, 2\pi)
\]

so that \( \log z \) is holomorphic on \( V = \mathbb{C} \setminus \{ x \mid x \geq 0 \} \).

\( f(z) \) has isolated singularities at the cube roots of \(-1,\)

\[ e^{\pi i/3}, \quad e^{2\pi i/3}, \quad e^{3\pi i/3} \quad \text{and} \quad e^{5\pi i/3}. \]
These are all simple singularities. As they all have modulus one the logarithm is purely imaginary at these points. We compute the residues:

$$\text{Res}_{e^{\pi i/3}} f(z) = \lim_{z \to e^{\pi i/3}} \frac{\log z}{3z^2} = \frac{\pi i}{3e^{2\pi i/3}} = \frac{\pi ie^{-2\pi i/3}}{9}.$$  

We also have

$$\text{Res}_{e^{3\pi i/3}} f(z) = \lim_{z \to e^{3\pi i}} \frac{\log z}{3z^2} = \frac{\pi i}{3e^{2\pi i}} = \frac{\pi i}{3},$$

and

$$\text{Res}_{e^{5\pi i/3}} f(z) = \lim_{z \to e^{5\pi i/3}} \frac{\log z}{3z^2} = \frac{5\pi i/3}{3e^{10\pi i/3}} = \frac{5\pi ie^{-4\pi i/3}}{9}.$$  

The residue theorem implies that

$$\int_{\gamma} \frac{\log z}{z^3 + 1} \, dz = 2\pi i \left( \text{Res}_{e^{\pi i/3}} f(z) + \text{Res}_{e^{3\pi i/3}} f(z) + \text{Res}_{e^{5\pi i/3}} f(z) \right)$$

$$= 2\pi i \left( \frac{\pi i}{9} \left( e^{-2\pi i/3} + 3 + 5e^{-4\pi i/3} \right) \right)$$

$$= \frac{2\pi i}{9} \left( \frac{-\sqrt{3}i}{2} + 3 - \frac{5}{2} + \frac{5\sqrt{3}i}{2} \right)$$

$$= -2\pi i \frac{2\pi}{3\sqrt{3}}.$$

Next we show the integrals over $\gamma_2$ and $\gamma_0$ go to zero. As usual we have to estimate the largest value of $|f(z)|$. Over $\gamma_2$ we have

$$|f(z)| = \left| \frac{\log z}{z^3 + 1} \right| \leq \frac{\ln R + 2\pi}{R^3 - 1}.$$
Thus
\[ \left| \int_{\gamma_2} \frac{\log z}{z^3 + 1} \, dz \right| \leq LM \]
\[ \leq \frac{2\pi R(\ln R + 2\pi)}{R^3 - 1}, \]
which goes to zero as \( R \) goes to infinity. Over \( \gamma_0 \) we have
\[ |f(z)| = \frac{|\log z|}{|z^3 + 1|} \]
\[ \leq \frac{2\pi - \ln \rho}{1 - \rho^3}. \]
Thus
\[ \left| \int_{\gamma_2} \frac{\log z}{z^3 + 1} \, dz \right| \leq LM \]
\[ \leq \frac{2\pi \rho(2\pi - \ln \rho)}{1 - \rho^3}, \]
which goes to zero as \( \rho \) goes to zero, since \( \rho \ln \rho \) goes to zero.

The integral over \( \gamma_+ \) is equal to
\[ \int_{\gamma_+} \frac{\log z}{z^3 + 1} \, dz = \int_{\rho}^{R} \frac{\ln x}{x^3 + 1} \, dx. \]

Finally, for the integral over \( \gamma_- \) we use the same parametrisation
\[ z = x \quad \text{where} \quad x \in [\rho, R] \]
but with a different branch of the logarithm
\[ \log z = \ln x + 2\pi i. \]
This traverses \( \gamma_- \) in the wrong direction, so we flip the sign.
\[ \int_{\gamma_-} \frac{\log z}{z^3 + 1} \, dz = - \int_{\rho}^{R} \frac{\ln x}{x^3 + 1} \, dx - 2\pi i \int_{\rho}^{R} \frac{dx}{x^3 + 1}. \]
Letting \( \rho \) go to zero and \( R \) go to infinity we get:
\[ -2\pi i I = -2\pi i \frac{2\pi}{3\sqrt{3}}. \]
Solving for \( I \) gives
\[ \int_{0}^{\infty} \frac{dx}{x^3 + 1} = \frac{2\pi}{3\sqrt{3}}. \]