# MODEL ANSWERS TO THE ZEROTH HOMEWORK

1. We have,

$$\begin{aligned}\frac{e^{2z}\sin(5z)}{1-z} &= \left(1+2z+2z^2+\dots\right)\left(5z-\frac{125}{6}z^3+\dots\right)\left(1+z+z^2+\dots\right)\\ &= 0+5z+15z^2+\left(10+10+5-\frac{125}{6}\right)z^3+\dots\\ &= 0+5z+15z^2+\frac{25}{6}z^3+\dots.\end{aligned}$$

2.

$$z \longrightarrow \frac{e^{2z}\sin(5z-1)}{1-z}$$

$e^{2z}$, $\sin(5z-1)$ and $1-z$ are all holomorphic, as the last function is a polynomial and the first two functions are the composition of a polynomial and a holomorphic function. The product of two holomorphic functions is holomorphic and the quotient of two holomorphic functions is holomorphic, away from the zeroes of the denominator.

The proof of Cauchy's formula shows that a function has a power series expansion based at any point where it is holomorphic. Moreover the function is not holomorphic somewhere on the circle defined by the radius of convergence. Since the only point where the function is not holomorphic is at 1, the point 1 must be on this circle. The distance of $a$ to 1 is $|a-1|$ and so this is the radius of convergence.

3. Suppose we try

$$z \longrightarrow \frac{az+b}{cz+d}.$$

As $\infty$ goes to $-1$, we have

$$\frac{a}{c}=-1 \qquad \text{so that} \qquad c=-a.$$

Thus we are reduced to

$$z \longrightarrow \frac{az+b}{d-az}.$$

As 0 goes to $i$, we have

$$\frac{b}{d}=i \qquad \text{so that} \qquad b=id.$$

Thus we are reduced to

$$z \longrightarrow \frac{az+id}{d-az}.$$

As 1 goes to 1 we have

$$\frac{a+id}{d-a} = 1 \qquad \text{so that} \qquad a+id = d-a$$

It follows that

$$2a = d(1-i).$$

If we put $d = 2$ then $a = 1 - i$. Thus we want

$$z \longrightarrow \frac{(1-i)z + 2i}{2 + (i-1)z}.$$

The points 0, 1 and $\infty$ belong to the real line. They are sent to three points of the unit circle, 1, $i$ and $-1$. Thus the real line is mapped to the unit circle.
The upper half plane is sent to to one of

- $\Delta$, the open unit disk, $|z| < 1$, or
- the complement of the closed unit disk $|z| > 1$.

But $i$ is sent to

$$\frac{(1-i)i + 2i}{2 + (i-1)i} = \frac{1+3i}{1-i}.$$

The number $1 + 3i$ is further from 0 than $1 = i$ so the point $i$ gets sent to a point of modulus bigger than one.
The upper half plane $\mathbb{H}$ is sent to the exterior of the unit circle.
4. (i) We use the paramaterisation

$$z = a + re^{i\theta}.$$

In this case

$$\mathrm{d}z = ire^{i\theta}\,\mathrm{d}\theta.$$

It follows that

$$\begin{aligned}\oint_{|z-a|=r} (z-a)^m \,\mathrm{d}z &= \int_0^{2\pi} r^m e^{mi\theta} ire^{i\theta}\,\mathrm{d}\theta \\ &= ir^{m+1}\int_0^{2\pi} e^{(m+1)i\theta}\,\mathrm{d}\theta.\end{aligned}$$

So are down to calculating the definite integral:

$$\int_0^{2\pi} e^{(m+1)i\theta}\,\mathrm{d}\theta.$$

This caculation breaks into two cases. Suppose that $m + 1 \neq 0$. Then the integral is zero. Indeed the anti-derivative of $e^{(m+1)i\theta}$ is $\frac{1}{m+1}e^{(m+1)i\theta}$ and when we calculate this at the two endpoints 0 and $2\pi$ we get the same answer, so that the difference is zero.

This argument breaks down if $m+1=0$. In this case we are integrating 1 over the interval $[0,2\pi]$ and the answer is $2\pi$. In this case $ir^{m+1}=i$. Thus

$$\oint_{|z-a|=r}(z-a)^m\,\mathrm{d}z=\begin{cases}2\pi i & \text{when } m=-1\\ 0 & \text{otherwise.}\end{cases}$$

(ii) Suppose that $m\geq 0$. Then $(z-a)^m$ is holomorphic on the closed disk $|z-a|\leq r$ and so Cauchy's theorem implies that

$$\oint_{|z-a|=r}(z-a)^m\,\mathrm{d}z=0.$$

Suppose that $m=-1$. Then we are integrating

$$\frac{f(z)}{z-a}=\frac{1}{z-a}\qquad\text{so that}\qquad f(z)=1.$$

Cauchy's integral formula implies that

$$\begin{aligned}1&=f(a)\\&=\frac{1}{2\pi i}\oint_{|z-a|=r}\frac{1}{z-a}\,\mathrm{d}z.\end{aligned}$$

Thus

$$\oint_{|z-a|=r}(z-a)^m\,\mathrm{d}z=2\pi i.$$

Now suppose that $m\leq -2$. Then we apply Cauchy's formula. If we differentiate $f(z)=1$ we get zero. Therefore

$$\begin{aligned}0&=f^{(-m-1)}(a)\\&=\frac{1}{2\pi i}\oint_{|z-a|=r}\frac{1}{(z-a)^{-m}}\,\mathrm{d}z.\end{aligned}$$

5.

$$\frac{1}{(z^2-1)(z^2-9)}$$

has isolated singularities at $\pm 1$ and $\pm 3$. There are two relevant circles centred at 0, the circle of radius 1, which contains the singularities $\pm 1$ and the circle of radius 3, which contains the singularities $\pm 3$.
These two circles divide the complex plane into three annuli,

$$\begin{aligned}U_0&=\{\,z\in\mathbb{C}\,|\,|z|<1\,\}=\Delta\\U_1&=\{\,z\in\mathbb{C}\,|\,1<|z|<3\,\}\\U_2&=\{\,z\in\mathbb{C}\,|\,3<|z|\,\}.\end{aligned}$$

To each annulus there is an associated Laurent expansion. For $U_0$ we want a power series centred at 0. We have

$$\begin{aligned}\frac{1}{(z^2-1)(z^2-9)} &= \frac{1}{1-z^2}\frac{1/9}{1-z^2/9}\\ &= \frac{1}{9}(1+z^2+z^4+z^6+\dots)\left(1+\frac{z^2}{9}+\frac{z^4}{81}+\dots\right)\\ &= \frac{1}{9}+\frac{10}{81}z^2+\frac{1}{9}\left(1+\frac{1}{9}+\frac{1}{81}\right)z^4+\dots.\end{aligned}$$

For $U_2$ we want a series centred at 0 that converges at $\infty$. We have

$$\begin{aligned}\frac{1}{(z^2-1)(z^2-9)} &= \frac{1/z^2}{1-1/z^2}\frac{1/z^2}{1-9/z^2}\\ &= \frac{1}{z^4}\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}+\dots\right)\left(1+\frac{9}{z^2}+\frac{81}{z^4}+\dots\right)\\ &= \frac{1}{z^4}+\frac{10}{z^6}+91\frac{1}{z^8}+\dots.\end{aligned}$$

For $U_1$ we want a Laurent series. We could try multiplying two series as above

$$\begin{aligned}\frac{1}{(z^2-1)(z^2-9)} &= -\frac{1/z^2}{1-1/z^2}\frac{1/9}{1-(z/3)^2}\\ &= -\frac{1}{9z^2}\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}+\dots\right)\left(1+\frac{z^2}{9}+\frac{z^4}{81}+\dots\right).\end{aligned}$$

However at this point we are stuck. It is not possible to make sense of multiplying the two series above algebraically (e.g. to calculate the constant term you would need to take a term of the form $1/z^{2m}$ from the first bracket and a term of the form $z^{2m+2}$ from the second bracket and there are infinitely many such terms). One can make sense of this analytically, supposing the coefficients form an absolutely convergent series, but this doesn't like a sensible way to proceed.
Instead we first use partial fractions to simplify the situation,

$$\begin{aligned}\frac{1}{(z^2-1)(z^2-9)} &= -\frac{1}{8(1-z^2)}+\frac{1}{8(z^2-9)}\\ &= -\frac{1/8z^2}{1-1/z^2}-\frac{1/72}{1-z^2/9}\\ &= -\frac{1}{8z^2}\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}+\dots\right)-\frac{1}{9}\left(1+\frac{z^2}{9}+\frac{z^4}{81}+\dots\right).\end{aligned}$$

6.
$$\frac{ze^z}{z^2-1}$$
has isolated singularities at $\pm 1$. They are both simple poles.
7.
$$f(z) = \frac{\sin z}{z^2}$$
has a simple pole at zero. It follows that
$$\begin{aligned}
\operatorname{Res}_0 \frac{\sin z}{z^2} &= \operatorname{Res}_0 f(z) \\
&= \lim_{z\to 0} z f(z) \\
&= \lim_{z\to 0} \frac{\sin z}{z} \\
&= \lim_{z\to 0} \frac{\cos z}{1} \\
&= 1.
\end{aligned}$$
Here we applied L'Hôpital's rule to get from the third line to the fourth line.

**Challenge Problems:** (Just for fun)
8. We first start by writing down examples.
$$z \longrightarrow z+1$$
has one fixed point. It does not fix any complex number but it does fix $\infty$.
$$z \longrightarrow \frac{1}{z}$$
has two fixed points. It fixes $\pm 1$. It switches 0 and $\infty$ and it is not hard to see that if $z$ is a complex number and
$$z = \frac{1}{z}$$
then $z = \pm 1$.
$$z \longrightarrow z$$
fixes everything. The only map with this property is the identity.
Now we turn to the general problem. First note that if $M$ fixes infinity then $c = 0$ so that $M(z)$ reduces to
$$z \longrightarrow az+b$$
If
$$z = az+b \qquad \text{then} \qquad (a-1)z = b.$$

If $a \neq 1$ then this equation has one solution and if $a \neq 1$ then it has no solutions, unless $b = 0$ in which case it is the identity.
So we may assume it does not fix $\infty$. In this case $c \neq 0$ and we may assume $c = 1$. For a fixed point we have

$$z = \frac{az+b}{z+d} \qquad \text{so that} \qquad z(z+d) = az + b.$$

Expanding we get

$$z^2 + (d-a)z + b = 0.$$

This is a quadratic equation for $z$. So it has one or two solutions and there are one or two fixed points.
9. Note that as $U$ is open and connected it is path connected. Fix $z_0$. Given $z \in U$ pick a path $\gamma$ from $z_0$ to $z$ and define

$$F(z) = \int_\gamma f(z)\,\mathrm{d}z.$$

The problem with this definition is that the integral might depend on the path.
Suppose that $\gamma_1$ and $\gamma_2$ are two paths connecting $z_0$ to $z$. Then

$$\gamma = \gamma_1 - \gamma_2$$

is a closed path starting and ending at $z_0$. We first go along $\gamma_1$ from $z_0$ to $z$ and then we go back along $\gamma_2$ from $z$ to $z_0$. By hypothesis

$$\begin{aligned} 0 &= \int_\gamma f(z)\,\mathrm{d}z \\ &= \int_{\gamma_1-\gamma_2} f(z)\,\mathrm{d}z \\ &= \int_{\gamma_1} f(z)\,\mathrm{d}z - \int_{\gamma_2} f(z)\,\mathrm{d}z. \end{aligned}$$

Thus

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

It follows that the definition of $F(z)$ does not depend on the path, so that we may write

$$F(z) = \int_{z_0}^{z} f(z)\,\mathrm{d}z.$$

Now we calculate the derivative of $F(z)$. By definition

$$F'(a) = \lim_{z \to a} \frac{F(z) - F(a)}{z - a}.$$

We start with the numerator:

$$\begin{aligned} F(z) - F(a) &= \int_{z_0}^{z} f(z)\,\mathrm{d}z - \int_{z_0}^{a} f(z)\,\mathrm{d}z \\ &= \int_{a}^{z} f(z)\,\mathrm{d}z. \end{aligned}$$

If $z$ is close enough to $a$ then we can draw a straight line from $a$ to $z$. If we paramaterise this we get

$$\gamma(t) = a + t(z-a) \qquad \text{where} \qquad t \in [0,1].$$

We get

$$\begin{aligned} \int_{a}^{z} f(z)\,\mathrm{d}z &= \int_{0}^{1} (z-a) f(a + t(z-a))\,\mathrm{d}t \\ &= (z-a) \int_{0}^{1} f(a + t(z-a))\,\mathrm{d}t. \end{aligned}$$

Thus

$$\begin{aligned} F'(a) &= \lim_{z \to a} \frac{F(z) - F(a)}{z - a} \\ &= \lim_{z \to a} \int_{0}^{1} f(a + t(z-a))\,\mathrm{d}t \\ &= \int_{0}^{1} \lim_{z \to a} f(a + t(z-a))\,\mathrm{d}t \\ &= \int_{0}^{1} f(a)\,\mathrm{d}t \\ &= f(a). \end{aligned}$$

Thus $f(z)$ is the derivative of $F(z)$.
It follows that $F(z)$ is holomorphic. As $F(z)$ is holomorphic it is infinitely differentiable. But then $f(z)$ is holomorphic.
10. We have

$$\begin{aligned} \int_{\gamma} f(z)\,\mathrm{d}z &= \int_{\gamma} \lim_{n \to \infty} f_n(z)\,\mathrm{d}z \\ &= \lim_{n \to \infty} \int_{\gamma} f_n(z)\,\mathrm{d}z \\ &= \lim_{n \to \infty} 0 \\ &= 0. \end{aligned}$$

To get from the first line to the second line we used uniform continuity and to get from the second line to the third line we use Cauchy's theorem. Morera's theorem implies that $f(z)$ is holomorphic.
Now we apply Cauchy's formula:

$$\begin{aligned} f'(a) &= \frac{1}{\pi i} \int_{|z-a|=r} \frac{f(z)}{(z-a)^2} \, \mathrm{d}z \\ &= \frac{1}{\pi i} \int_{|z-a|=r} \lim_{n\to\infty} \frac{f_n(z)}{(z-a)^2} \, \mathrm{d}z \\ &= \lim_{n\to\infty} \frac{1}{\pi i} \int_{|z-a|=r} \frac{f_n(z)}{(z-a)^2} \, \mathrm{d}z \\ &= \lim_{n\to\infty} f_n'(a). \end{aligned}$$

This gives pointwise convergence of the derivative. Uniform convergence follows from standard results about uniform convergence (which we won't prove in this course).